Global Teacher Empowerment Network GTEN
THE GOLDEN RATIO IN MATHS, SCIENCE AND ART
Saturday 24 September 2022   16:00 – 18:00 London

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AIMS | African Institute for Mathematical Sciences — SCHOOLS ENRICHMENT CENTRE

MATHS TOYS

Global Teacher Empowerment Network (GTEN)
PROGRAMME: THE GOLDEN RATIO IN MATHEMATICS SCIENCE AND ART

IMPROVE SKILLS, KNOWLEDGE AND UNDERSTANDING OF:

- Number patterns and sequences
- Ratios
- Spirals
- Geometric constructions
- Similar and congruent triangles
- Properties of a regular pentagon
- Quadratic Equations
- Connections between mathematics, art, architecture and science.

LEARNING SPIRAL

UPPER SECONDARY

LOWER SECONDARY

UPPER PRIMARY

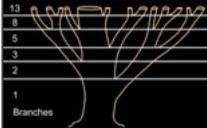**LOWER PRIMARY**

EARLY YEARS

12. The Golden Ratio by square roots and by continued fractions.
11. Euler's definition of the Golden Ratio.
10. Solving the quadratic equation that gives the Golden Ratio.
9. Construction of the Golden Rectangle by ruler and compass and by paper folding.
8. The Golden Ratio in a Regular Pentagon.
7. Examples of the Golden Ratio in Art, Architecture and Science
5. The ratios of successive terms in the Fibonacci Sequence
4. The Elephant Dreaming spiral
2. The Sheep Talk number sequence
1. Starter activity: tie a knot in a strip of paper making a pentagon

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During this session you need to wear 2 hats to appreciate how a learner at a particular stage would do the activity and what they would learn from it.

Teacher

Learner

DO-TALK-RECORD
the basis of a good lesson

DO the activities pretending that you only know what your learners know.

TALK to other teachers in your group, or comment on the chat. How might learners discover new ideas by doing the activity?

RECORD for yourself as a teacher and also in a way that would help learners.

DO   Count branches at each level.
   TALK   What do you notice?
      RECORD   Make notes.
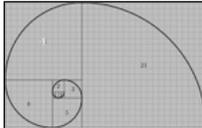
DO   Study this spiral.
TALK   What do you notice?
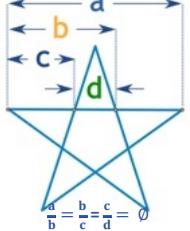RECORD   Copy the spiral accurately.

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The Pentagram and The Golden Ratio φ

DO: Take a strip of paper and tie a knot, then flatten it carefully. Hold it up to the light. Can you see the pentagon with the five pointed star or pentagram inside?

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \phi$$

The pentagram star symbol appears in many cultures and in some of the flags of the world.

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Slide 5

The Golden Rectangle and The Golden Ratio ϕ

x -1   1

1   1

1

x

The ratios of the long to short edges of the Golden Rectangle are $\frac{x}{1}$ and $\frac{1}{x-1}$

The Golden Ratio is given by solving the equation $\frac{x}{1} = \frac{1}{x-1}$

The Golden Rectangle has the property that it can be divided into a rectangle and a square where **the inner rectangle has the same proportions as the outer rectangle.**

The ratio of the long to short edges is the same.

This process of splitting the rectangle into a square and a smaller rectangle of the same proportions can be repeated again and again indefinitely.

The Golden Ratio has been observed and studied for thousands of years. It has influenced designs in architecture and art.

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Slide 6

SHEEP TALK

Sheep Talk The first sheep talk word is **A**. To find the next term in the word sequence, each A is replaced by B and each B is replaced by AB.

The rule is: **A -> B and B-> AB**

Here are the first 7 words in the sheep talk sequence:

A B AB BAB ABBAB BABABBAB ABBABBABABBAB

DO: Count the number of occurrences of A, of B, and the total number of letters, and fill in the table. Spot the pattern in the sequences and TALK on chat?

Number of A's										
Number of B's										
Number of letters										

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Slide 7

SHEEP TALK

Sheep Talk The rule is: **A -> B and B-> AB**

Here are the first 7 words in the sheep talk sequence:

A B AB BAB ABBAB BABABBAB ABBABBABABBAB

Number of A's	1	0	1	1	2	3	5	8	13	21	34
Number of B's	0	1	1	2	3	5	8	13	21	34	55
Number of letters	1	1	2	3	5	8	13	21	34	55	89

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Slide 8

ELEPHANT DREAMING FIBONACCI SPIRAL

DO: Copy the picture below on squared paper, draw the spiral and create your own elephant.

The numbers in the centres of the squares give the radii of the quarter circles you need to draw. For the elephant's trunk, start from the quarter circle of radius 3 and don't draw the quarter circles with radii 1, 1, and 2.

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The Fibonacci Sequence

The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …
What is the rule for finding the next term in the sequence?

The ratios of the terms 1, 2/1, 3/2, 5/3, 8/5, 13/8 … are the ratios of the edge lengths of the rectangles in the Elephant Dreaming construction of the Fibonacci Spiral.

Write these fractions as decimals and continue these sequences. What do you notice?

FIBONACCI SQUARES

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The Fibonacci Sequence

Did you notice that there is a repeating pattern of rectangles being split

into a square and another rectangle again and again at smaller scales

and the edge lengths of the rectangles are the Fibonacci numbers?

Unlike the Golden Rectangle the proportions do not stay the same

as the ratios of the edge lengths of the rectangles

are the ratios of successive terms of the Fibonacci sequence.

FIBONACCI SQUARES

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The Fibonacci Sequence and The Golden Ratio

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …
The ratios of successive terms are:
1, 2, 1.5, $1.\overline{6}$, 1.6, 1.625, 1.615, 1.619, 1.618…
These ratios converge to The Golden Ratio ϕ.

FIBONACCI SQUARES

Did you notice that adding two successive terms in the Fibonacci sequence gives the next term in the sequence
1 + 1 = 2, 1 +2 = 3, 2 + 3 = 5 etc. ?

The sequence is defined, giving the nth term as F_n , by the formula:

$$F_n = F_{n-1} + F_{n-2}$$ with first two terms $F_0=1$ and $F_1=1$

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WHAT TO YOU NOTICE?

Lily   Euphorbia   Trilium   Columbine   Daisy

13
8
5
3
2
1
Branches

Pinecone   Sunflower   Nautilus shell

12

3

Slide 13

The Golden Ratio appears in Art, Architecture and Nature

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Slide 14

https://www.geogebra.org/m/B4C9bbuv

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Slide 15

GEODESIC DOMES

Regular pentagons and hexagons form the Buckminster fullerene or bucky ball, as in soccer balls. Geodesic Domes can be built with the same configuration using irregular polygons.

Geodesic Domes are energy efficient. Compared to other structures, light reflects and stays in the dome longer, the high volume to surface area ratio means less materials are used in construction, and heating costs are low.

Bucky ball

Eden Project Biome Greenhouses

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Slide 16

**The Great Golden Pyramid of Giza c. 2560 BC**

The structure consists of approximately 2 million blocks of stone, weighing 6 million tonnes in total.

The Great Pyramid, with dimensions: 230 m by 230 m and height 146.6 m, is bigger than any of the great cathedrals. For example it's bigger than Milan Cathedral which holds 40,000 people.

St Peter's, Rome   St Paul's, London   Westminster Cathedral

Milan Cathedral   Florence Cathedral

**Man fears Time,
yet Time fears the Pyramids**
Arab proverb

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The Divine Proportion in Art and Nature
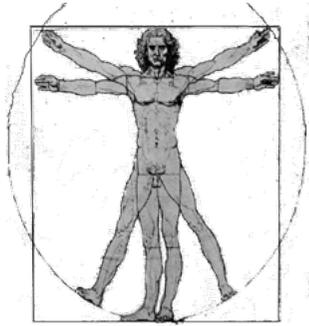
In this drawing by Leonardo da Vinci the height of a person is divided into two sections, the dividing point being the navel. The distance from the navel to the the soles of the feet divided by the distance from the top of the head to the navel was equal to 1.618, the golden ratio.

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Salvador Dali The Last Supper 1955
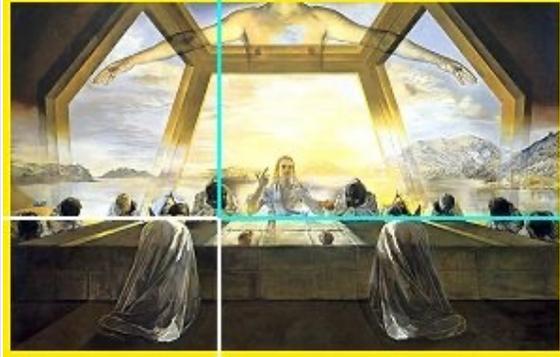
Salvador **Dali** 1904-1989 "The Sacrament of the **Last Supper**". National Gallery of Art, Washington DC. 1955

The painting shows part of a dodecahedron. The golden ratio occurs in many ways in regular pentagons and regular dodecahedra.

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The Golden Ratio is also called The Divine Proportion or Golden Mean or Golden Section or Golden Number

Leonardo of Pisa (Fibonacci) was the son of a merchant and travelled widely, meeting with many merchants and learning about their methods of doing arithmetic. He realised the many advantages of the Hindu-Arabic number system, which, unlike the Roman numerals used at the time, allowed easy calculation using a place-value system.

His book Liber Abaci (*Book of Abacus* or *The Book of Calculation*), popularized Hindu–Arabic numerals and revolutionised calculation methods in Europe.

In this book Leonardo also wrote about the the Golden Ratio and the sequence which is named after him.
 https://en.wikipedia.org/wiki/Fibonacci

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Fibonacci's Rabbits Problem

Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Young rabbits mate at the age of one month, so that at the end of its second month a female is mature and can produce another pair of rabbits.

Suppose that our rabbits have long lives and that the female always produces one new pair (one male, one female) every month from the second month on.
How many rabbits will there be at the end of 3 months?
How many at the end of one year?

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Fibonacci's Rabbits

The diagram shows the population of rabbits at times t = 0, 1, 2, 3, 4 and 5 months.

At time t = 0 the original pair of new-born rabbits are shown by a yellow disc.

At time t = 1 month the same pair are shown in pink, as young rabbits.

At time t = 2 months the diagram shows the same rabbits (now mature) in red together with a pair of new-borns (in yellow).

At time t = 3 months the original pair have produced a 2nd pair of new-borns (orange) and their 1st pair are now young and shown in pink.

The population is also shown at times t = 4 and 5 months.

Extend the tree diagram to show the population at t = 6 months and fill in the table.

Fibonacci's Rabbits

Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits mate at the age of one month, so that at the end of its second month a female can produce another pair of rabbits.

Suppose that our rabbits have long lives and that the female always produces one new pair (one male, one female) every month from the second month on.

What do you notice?

Time in months	0	1	2	3	4	5	6	7	8	9	10	11	12
Number of new-born pairs N_t	1	0	1	1	2	3	5	8	13	21	34	55	89
Number of young pairs Y_t	0	1	0	1	1	2	3	5	8	13	21	34	55
Number of mature pairs M_t	0	0	1	1	2	3	5	8	13	21	34	55	89
Total number of pairs F_t	1	1	2	3	5	8	13	21	34	55	89	144	233

The Golden Rectangle and The Golden Ratio ϕ

The ratios are: $\dfrac{x}{1} = \dfrac{1}{x-1}$

This leads to the equation $x^2 - x - 1 = 0$

Solve this equation.

The solutions are: $(1 \pm \sqrt{5})/2$

The positive solution $x = (1 + \sqrt{5})/2$

is the Golden Ratio which is denoted by

the Greek letter $\emptyset$ = 1.618 to 4 s.f.

Golden Ratio ϕ

Euclid in *The Elements*, says that the point C divides the line *AB* in the Golden Ratio if and only if the extreme ratio AB : AC is equal to the mean ratio AC : CB

$AB : AC = AC : CB$

Similarly $BA:BC_1 = BC_1:C_1A$

If $AB = x$ units and $AC = 1$ unit

then $\qquad AB : AC = AC : CB$

becomes $\qquad x : 1 = 1 : x - 1$

or $\qquad \dfrac{x}{1} = \dfrac{1}{x-1}$ **Do you recognize this equation?**

This leads to the equation $x^2 - x - 1 = 0$.

We have solved this equation. The solutions are: $(1 \pm \sqrt{5})/2$

The positive solution $x = (1 + \sqrt{5})/2$ is called the Golden Ratio

denoted by the Greek letter $\emptyset$

Slide 25

The Golden Ratio (or Divine Proportion) in Art and Nature

The Parthenon in Athens, built in 447 BC, is an example of the use of the golden ratio in architecture.

http://academic.reed.edu/humanities/110Tech/Parthenon.html

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Slide 26

The Last Supper   Leonardo da Vinci 1452-1519 Convent of Santa Maria delle Grazie, Milan

The Golden Ruler, and the markings on the painting, show the two, symmetrically placed, positions of the golden section points.

The coloured sections on the golden ruler show the section points as the rectangles are split into squares and rectangles at smaller and smaller scales.

The Golden Ruler

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Slide 27

The Pentagram and The Golden Ratio ϕ

The ratio of the chord length to the edge length in a regular pentagon is the Golden Ratio ϕ

Test this by measuring AC and AB and calculating $\dfrac{AC}{AB}$

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Slide 28

The Regular Pentagon and The Golden Ratio ϕ

https://aiminghigh.aimssec.ac.za/golden-pentagon/

Measure the red and blue lengths in each of these pictures and calculate the ratio of the red to the blue lengths in each case. each of the ratios is equal to the Golden Ratio.

Proofs of these results involve similar triangles.

List triangles 1 to 10 grouping them into sets of congruent triangles

Can you spot all the similar triangles?
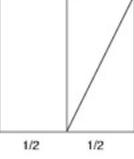
Can you work out all the angles?

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Slide 29

A construction for the Golden Rectangle

1. Draw a square of side 1 unit near the left side of the paper (10 cm is a convenient length)
2. Bisect the base
3. Draw the diagonal to form a triangle with sides 1 unit and 1/2 unit so that the diagonal has length $\sqrt{5}/2$
4. Draw the arc BE of the circle centre M with radius MB and find the point E where this arc cuts the extended base to give one vertex of the golden rectangle AFED.
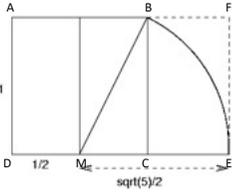
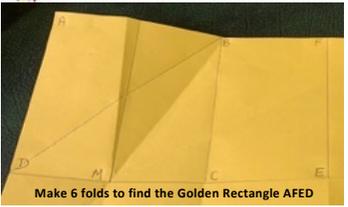As ME = $\sqrt{5}/2$ the edges of AFED are $(1+\sqrt{5})/2$ and 1 (ratio ϕ:1) so AFED is a Golden Rectangle.

If you have no compasses you can measure the distance MB but it won't be as accurate.
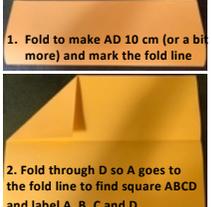
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Slide 30

The Golden Rectangle by paper-folding

Make 6 folds to find the Golden Rectangle AFED

1. Fold to make AD 10 cm (or a bit more) and mark the fold line
2. Fold through D so A goes to the fold line to find square ABCD and label A, B, C and D.
3. Fold AD to BC to find the midpoint M of DC.
4. Fold along BM putting A at the back
5. To find E, fold through M to take B to line DMC
6. To find F, make a fold through E with the top edge folding back on itself.

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Slide 31

$$x = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$$

What do you notice in this formula?

$$x^2 = 1 + \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$$
$$= 1 + x$$

$$x^2 - x - 1 = 0 \qquad \text{We have already solved this equation.}$$

The solutions are: $(1 \pm \sqrt{5})/2 = (1+\sqrt{5})/2 = 1.618$ to 4 s.f.

The value of x given by this formula is equal to the Golden Ratio

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$$

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Slide 32

Continued Fractions

What do you notice about this sequence? x, $1+\dfrac{1}{x}$, $1+\dfrac{1}{1+\dfrac{1}{x}}$, $1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$, ...

Again, a repeating pattern where x is replaced by $1 + 1/x$. The pattern goes on and on for ever.

Suppose we choose a value of x and calculate the values of these fractions. Let's do this for $x = 2$

What do you notice?

$1+\dfrac{1}{2}=\dfrac{3}{2}$;

$1+\dfrac{1}{1+\dfrac{1}{2}}=1+\dfrac{1}{3/2}=1+\dfrac{2}{3}=\dfrac{5}{3}$;

$1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{2}}}=1+\dfrac{1}{5/3}=1+\dfrac{3}{5}=\dfrac{8}{5}$;

$1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{2}}}}=1+\dfrac{1}{8/5}=\dfrac{13}{8}$

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Slide 33 — Continued Fractions

$$x, \quad 1+\frac{1}{x}, \quad 1+\frac{1}{1+\frac{1}{x}}, \quad 1+\frac{1}{1+\frac{1}{1+\frac{1}{x}}}, \dots$$

Continued Fractions

We calculate the values of these fractions for $x = 2$

Think of each continued fraction as $1 + \frac{1}{x}$ and look at the colours. You will see that the denominator x corresponds to the continued fraction BEFORE it in the sequence.

For $x = 2$ the sequence is:

$$2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \dots$$

The continued fraction values give the sequence of ratios of the Fibonacci numbers that we saw earlier which converges to $\emptyset$ = **1.618 to 4 sf.**

$x = 2$

$$1 + \frac{1}{2} = \frac{3}{2};$$

$x = 3/2$

$$1 + \frac{1}{1+\frac{1}{2}} = 1 + \frac{1}{3/2} = 1 + \frac{2}{3} = \frac{5}{3};$$

$x = 5/3$

$$1 + \frac{1}{1+\frac{1}{1+\frac{1}{2}}} = 1 + \frac{1}{5/3} = 1 + \frac{3}{5} = \frac{8}{5};$$

$x = 8/5$

$$1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}} = 1 + \frac{1}{8/5} = \frac{13}{8}$$

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Slide 34 — SUMMARY

SUMMARY

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\dots}}}}$$

Great Pyramid of Giza

Sheep Talk
A   B   AB   BAB   ABBAB
BABABBAB   ABBABBABABBAB

Elephant Dreaming

Fibonacci sequence in generations of rabbits

1, 1, 2, 3, 5, 8, 13, 21, 34, 55 …

Fibonacci Spiral

PENTAGON WASHINGTON DC

GOLDEN PENTAGON

Golden Ratio by construction

1/2   sqrt(5)/2

FIBONACCI SQUARES

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Slide 35 — AIMINGHIGH TEACHER NETWORK

AIMINGHIGH TEACHER NETWORK

AIMSSEC

On the AIMING HIGH website there are freely downloadable worksheets, Notes for Teachers with solutions, Key Questions and Diagnostic Quizzes for formative assessment, also Inclusion and Home Learning Guides with activities for learners of all ages and attainment levels.

SHEEP TALK   https://aiminghigh.aimssec.ac.za/sheep-talk/
ELEPHANT DREAMING https://aiminghigh.aimssec.ac.za/elephant-dreaming/
GOLDEN PENTAGON   https://aiminghigh.aimssec.ac.za/golden-pentagon/
ONE STEP TWO STEPS https://aiminghigh.aimssec.ac.za/one-step-two-steps/
FIBONACCI'S RABBITS https://aiminghigh.aimssec.ac.za/fibonaccis-rabbits/
NATURE BY NUMBERS Cristobal Vila https://youtu.be/kkGeOWYOFoA/
 A SUMMARY FOR THE END OF A SEQUENCE OF LESSONS Ben Sparks https://youtu.be/z9d1mxgZ0ag

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Slide 36 — AIMS GTEN

AIMS | African Institute for Mathematical Sciences SCHOOLS ENRICHMENT CENTRE

Global Teacher Empowerment Network (GTEN)

For teachers in primary and secondary schools, colleges and universities

MATHS TOYS

AIMSSEC GTEN YouTube Channel
https://www.youtube.com/c/MathsToys/videos

AIMSSEC FACEBOOK https://www.facebook.com/aimssecsa/

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Thanks for coming to this workshop.

Use the AIMSSEC ideas
on AIMING HIGH and add comments.

Share what you have learned
with other teachers.

Try to help all your learners to have a

'YES I CAN'

attitude to mathematics.

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Caroline Ainslie caroline@bubblymaths.co.uk

Enquire about signing up for an AIMSSEC course
as a self-funding student admin@aimssec.ac.za

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