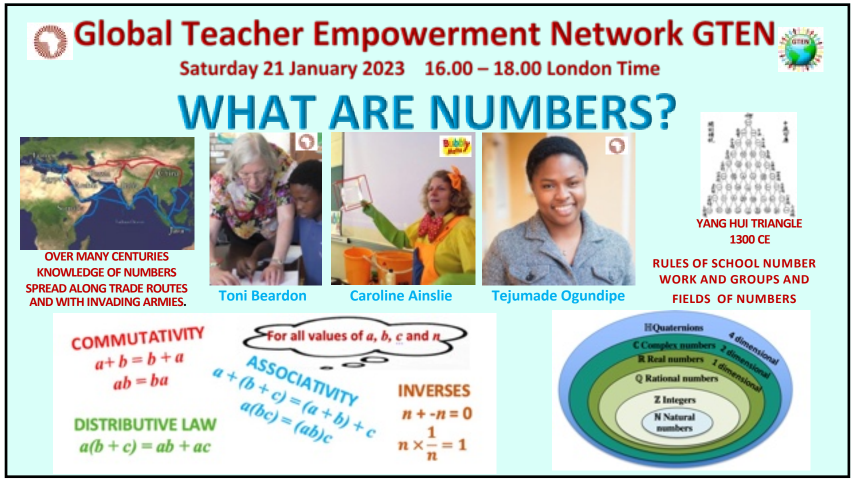
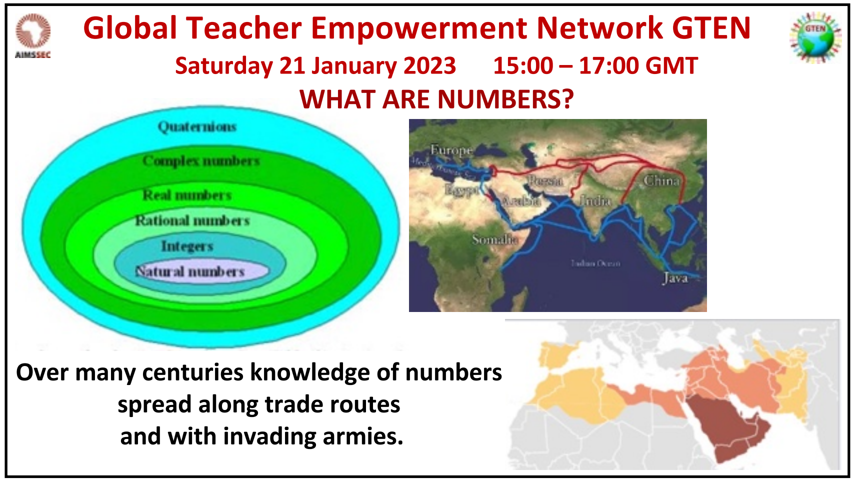
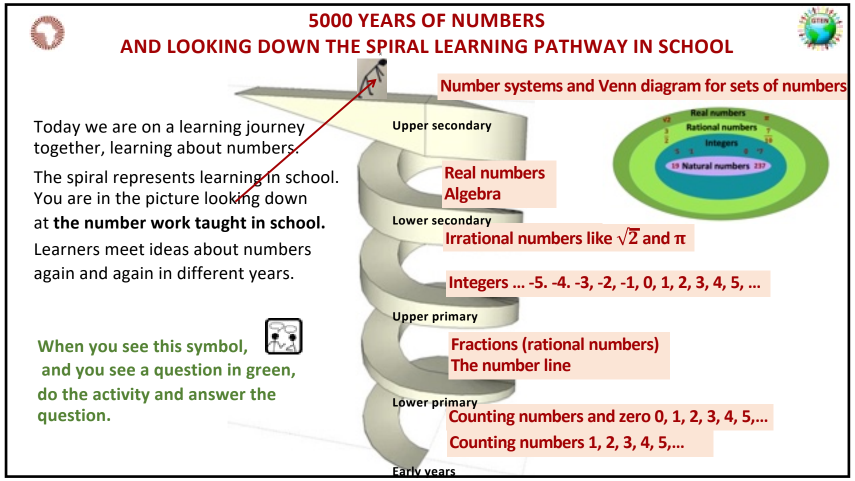

Global Teacher Empowerment Network GTEN
Saturday 21 January 2023 16.00 – 18.00 London Time
WHAT ARE NUMBERS?
OVER MANY CENTURIES KNOWLEDGE OF NUMBERS SPREAD ALONG TRADE ROUTES AND WITH INVADING ARMIES.
Toni Beardon
Caroline Ainslie
Tejumade Ogundipe
YANG HUI TRIANGLE 1300 CE
RULES OF SCHOOL NUMBER WORK AND GROUPS AND FIELDS OF NUMBERS
COMMUTATIVITY
$a+b=b+a$
$ab=ba$
For all values of a, b, c and n
ASSOCIATIVITY
$a+(b+c)=(a+b)+c$
$a(bc)=(ab)c$
INVERSES
$n+-n=0$
$n\times\frac{1}{n}=1$
DISTRIBUTIVE LAW
$a(b+c)=ab+ac$
Quaternions
Complex numbers
Real numbers
Rational numbers
Integers
Natural numbers

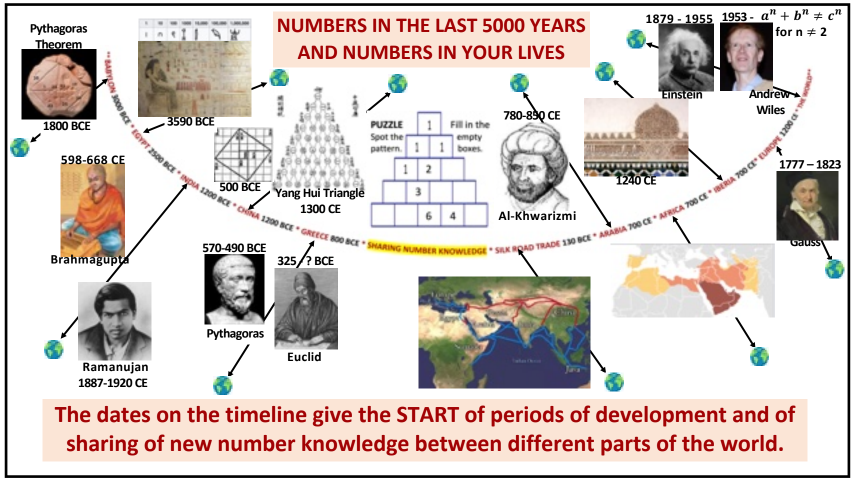
1


Global Teacher Empowerment Network GTEN
Saturday 21 January 2023 15:00 – 17:00 GMT
WHAT ARE NUMBERS?
Quaternions
Complex numbers
Real numbers
Rational numbers
Integers
Natural numbers
Europe
China
India
Java
Over many centuries knowledge of numbers spread along trade routes and with invading armies.

2

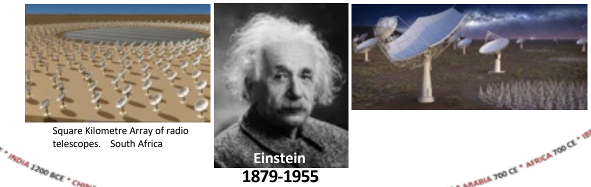

5000 YEARS OF NUMBERS
AND LOOKING DOWN THE SPIRAL LEARNING PATHWAY IN SCHOOL
Today we are on a learning journey together, learning about numbers.
The spiral represents learning in school. You are in the picture looking down at the number work taught in school.
Learners meet ideas about numbers again and again in different years.
When you see this symbol, and you see a question in green, do the activity and answer the question.
Number systems and Venn diagram for sets of numbers
Upper secondary
Real numbers
Algebra
Lower secondary
Irrational numbers like $\sqrt{2}$ and π
Integers ... -5. -4. -3, -2, -1, 0, 1, 2, 3, 4, 5, ...
Upper primary
Fractions (rational numbers)
The number line
Lower primary
Counting numbers and zero 0, 1, 2, 3, 4, 5,...
Counting numbers 1, 2, 3, 4, 5,...
Early years

3


NUMBERS IN THE LAST 5000 YEARS AND NUMBERS IN YOUR LIVES
Pythagoras Theorem
1800 BCE
3590 BCE
598-668 CE
Brahmagupta
500 BCE
Yang Hui Triangle 1300 CE
PUZZLE
780-850 CE
Al-Khwarizmi
1240 CE
1879 - 1955
Einstein
1953 - $a^n+b^n\neq c^n$ for $n\neq 2$
Andrew Wiles
1777 – 1823
Gauss
570-490 BCE
Pythagoras
325 / ? BCE
Euclid
Ramanujan 1887-1920 CE
The dates on the timeline give the START of periods of development and of sharing of new number knowledge between different parts of the world.

4

NUMBERS IN THE LAST 5000 YEARS AND NUMBERS IN YOUR LIVES



Today you review all the number work done in school from Early Years to School Leaving.
You will learn that school number work is actually about **groups and fields of numbers**.
You will meet the idea of groups and fields of transformations in space, and
touch on **4-dimensional numbers** with 3-spatial dimensions and time as the 4th dimension.
These number systems are the basis of a large part of Higher Mathematics and Science,
for example, in **Einstein field equations** for the **general theory of relativity**
and in studying the distribution of matter in the universe.

5

SILLY JOKE 1

Why don't you do calculations in a game reserve?

Answer:

Because if you add 4+4 you get ate!

6

NUMBERS IN THE LAST 5000 YEARS AND NUMBERS IN YOUR LIVES
including these topics from the S. African CAPS National Curriculum Statement

YEAR 9

Properties of numbers

• Describe the real number system by recognising, defining and distinguishing properties of:
- natural numbers
- whole numbers
- integers
- rational numbers
- irrational numbers

What do the highlighted words mean?

Properties of integers

• Recognise:
- Commutative, associative and distributive properties of addition and multiplication for integers
- Additive and multiplicative inverses for integers

• Solve problems in contexts involving multiple operations with integers

Use the theorem of Pythagoras
to calculate a missing length in a right-angled triangle, leaving irrational answers in surd form.

7

NUMBERS IN THE LAST 5000 YEARS AND NUMBERS IN YOUR LIVES
including these topics from the S. African CAPS National Curriculum Statement

YEARS 10 AND 11

Algebra

-Understand that real numbers can be irrational or rational.

-Take note that there exist numbers other than those on the real number line, the complex numbers.

It is possible to SQUARE certain complex numbers and obtain negative real numbers as answers.

-Nature of roots.

-Solve quadratic equations.

Examples:

- Show that the roots of $x^2 - 2x - 7 = 0$ are irrational.

- Show that $x^2 + x + 1 = 0$ has no real roots.

You are going to do some university level maths today including algebra with complex numbers. You will learn about 2-dimensional and 4-dimensional numbers and their uses in science.

8

THE SPIRAL LEARNING PATH – PUT YOUR LEARNER'S HAT ON

- When you were very small you learned to count, and you have learned more about numbers each year in school.
- Now you know about counting numbers, negative numbers, fractions and irrational numbers like $\sqrt{2}$ and π and the rules of +, -, ×, ÷ for combining them.
- Today we'll start with counting numbers and we'll see how people met problems they could not solve so they had to discover new types of numbers.
- We'll meet higher dimensional numbers and their uses in science.
- We'll meet school mathematics as a set of rules that must be followed exactly like the rules of any game. These rules also give us the idea of a group.
- Apart from numbers the group-rules apply to many other systems that are essential to higher mathematics, physics and chemistry.

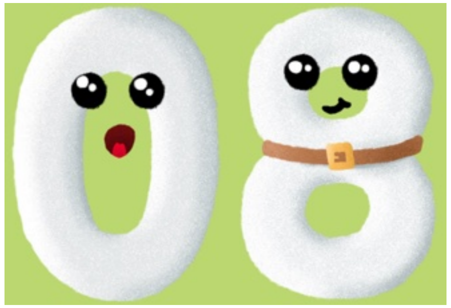
Let's get our ducks in a row!
Thank you Mike Pearson for the ducks.

9

SILLY JOKE 2

Question: What did zero say to 8

Answer: Nice belt



See https://learnfunnyjokes.com/funny-jokes?tag=Math%20jokes for more maths jokes.

10

THE QUESTION: "WHAT ARE NUMBERS?"

There are many different answers to this question.
Think about how you met different types of numbers when you were younger.

First you learned to count up to, say 100.
Do you remember when you realised that counting goes on for ever with no end?

Wow! The set of counting numbers 1, 2, 3, 4, 5, … is INFINITE.

The concept of infinity is very important.

11

THE RULES OF ARITHMETIC THAT YOU HAVE LEARNED

What would you put in the box?

5 + ☐ = 8

Yes, 3 goes in the box. Easy! You learned that in Year 1.

Early in your lives you met adding and sharing.

Adding and sharing take us beyond simple counting into number work (arithmetic).

At that stage you understood numbers are not just for counting.

Numbers are objects we combine according to agreed rules, which we call arithmetic.

The more we know about arithmetic, the more we see that it is a useful tool for solving all sorts of human problems.

12

FORMAL LANGUAGE FOR WHAT YOU LEARNED IN PRIMARY SCHOOL INVERSE OPERATIONS AND ZERO

Addition corresponds to increasing the size of a collection of objects by combining two collections.

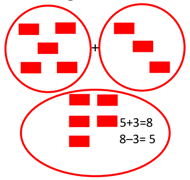
5 + 3 = 8

Can we undo addition?

8 - 3 = 5

Subtraction reduces the size by removing some of the objects.

Subtraction undoes addition.



What would this answer be?

5 - 5 = ☐ **0 goes in the box.**

If we use numbers to describe the size of a collection of objects, we need a number for a set that has nothing in it, **the empty set**. We need to know about **the number zero**.

Use of numbers dates back at least five thousand years, but as recently as five hundred years ago scholars in Europe were still asking "Is zero a number or not?"

13

Did you hear about the mathematician who was afraid of negative numbers?

He'd stop at nothing to avoid them.

14

THINK BACK TO THE TIME YOU DID NOT KNOW ABOUT NEGATIVE NUMBERS

When you only knew about the whole numbers 0, 1, 2, 3, 4, …., and not about negative numbers,

What could you put in these boxes?

8 + 3 = ☐

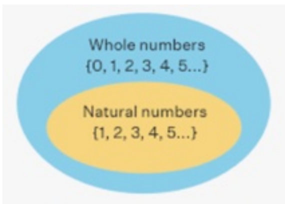
5 + ☐ **= 12**

8 - 3 = ☐

3 - 8 = ☐

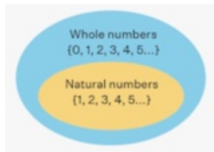
12 - 5 = ☐

5 - 12 = ☐



15

EXPLAIN THE MEANING OF THE WORD CLOSURE IN MATHEMATICS



When you only knew about the whole numbers 0, 1, 2, 3, 4, …., and not about negative numbers, what could you put in these boxes?

When you add two whole numbers you get a whole number.

ADDITION IS CLOSED FOR THE SET OF WHOLE NUMBERS

When you subtract one whole number from another you **DON'T** always get a whole number.

SUBTRACTION IS NOT CLOSED FOR THE SET OF WHOLE NUMBERS

8 + 3 = 11

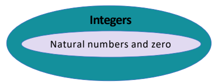
5 + 7 = 12

8 - 3 = 5

3 - 8 = ☐

12 - 5 = 7

5 - 12 = ☐



You need to learn about **negative numbers** and **integers** before you can find answers to put in these boxes.

16

Stand up

Sit down

Stand up

Sit down

You don't need to do this.

Just think about it.

Put your coat on

Take your coat off

Put your coat on

Take your coat off

What do you notice about these actions?

Would tying and untying shoelaces do and undo a process?

👍 for YES 👎 for NO

Explain your choice of yes or no.

17

Question:

Why won't Mzu drink a glass of water with 8 pieces of ice in it?

Answer:

It's too cubed.

$2^3 = 8$

18

EXPLAIN INVERSE OPERATIONS. WHAT ARE THEY?

Seeing numbers in your mind – mental arithmetic

Close your eyes and keep them closed.

Choose your own number and remember it

Make a mental picture of your number

Now add 4 to your number and think of the new number

With this new number subtract 4.

Think about what you have just done.

What happened?

Open your eyes.

Write down your final answer. Explain it?

Suppose you chose 11 to start

11 + 4 = 15 15 – 4 = 11

Addition and subtraction undo each other they are called INVERSE OPERATIONS

19

NEGATIVE NUMBERS AND INVERSE PAIRS

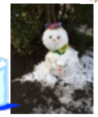
What is the answer to 9 – 13?

9 – 13 = ☐

When you did not know about negative numbers you thought there was no answer. You need -4.

What number do we add to 13 to make 9?

13 + ☐ = 9



Subtraction is just addition of the inverse. $(^{+}13) + (^{-}4) = 13 - 4 = {}^{+}9$

$9 - 13 = (^{+}9) + (^{-}13) = {}^{-}4$

$^{-}4$

... -10 -9 -8 -7 -6 -5 **-4** -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7 +8 **+9** +10 +11 +12 **+13** +14 ...

-13

The number minus 4 (written $^{-}4$) is called the inverse of $^{+}4$ because $(^{+}4) + (^{-}4) = 0$

20

EXPLAIN WHY + - AND - + MAKE -
WHY DOES - - MAKE + ?

Explain why $7 - {}^{-}4 = 7 + 4$ Two minuses make a plus

Subtraction is just addition of the inverse. $-({}^{-}4) = +({}^{+}4)$

$7 - ({}^{-}4) = 7 + ({}^{+}4) = {}^{+}11$

... -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7 +8 +9 +10 +11 +12 +13 +14 ...

$^{+}4$

The number **minus 4 (written $^{-}4$)** is called the inverse of $^{+}4$ because **$({}^{+}4) + ({}^{-}4) = 0$**

Every integer has **an inverse** such that the **number added to its inverse gives zero.**

This is a **number system** and we are using the **rules of school arithmetic.**

21

PUTTING NUMBERS IN BOXES IN PRIMARY SCHOOL
IS THE SAME AS SOLVING EQUATIONS IN SECONDARY SCHOOL

You have learned to solve equations of the form $a + x = b$ when $b > a$.

You add the inverse $-a$ to both sides and get $x = b - a$

When you did not know about negative numbers could you solve the equation $9 + x = 5$?

No because 9 > 5.

$9 + x = 5$ has no solution in the set of whole numbers,

but it has the solution $x = {}^{-}4$ in the set of integers.

For all equations like this to have solutions you need to work with **the set of integers**.

22

THE SAME PROPERTIES AGAIN FOR FRACTIONS OR RATIONAL NUMBERS

There was a time when you first learned about negative numbers,

and another time when you met fractions.

When you did not know about rational numbers (fractions) could you put numbers in all these boxes?

10 ÷ 5 = ☐

10 ÷ 2 = ☐

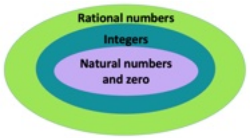
10 ÷ 3 = ☐

Without fractions, could you solve the equations?

$5x = 10$

$2x = 10$

$3x = 10$



Mathematicians call fractions **rational numbers,** for example $\frac{10}{3}$

When you +, -, × and ÷ rational numbers (except 0) you always get a rational number.

It's a closed system.

23

YOU HAVE PUT NUMBERS IN BOXES AND SOLVED EQUATIONS FOR ADDITION
NOW THINK ABOUT SOLVING EQUATIONS THAT INVOLVE MULTIPLICATION

If a, b and x are integers, a and b are known and we have to find x, can we solve ALL equations of the form $ax = b$?

for example can you solve $11x = 120$ for integers, YES or NO? 👍 **for YES** 👎 **for NO**

To solve equations you use the inverse operation (like untying shoelaces). In this example, when you know about fractions you can use the inverse operation, and divide by 11 to find the answer.

The answer is not an integer, instead it is a fraction (rational number) $\frac{120}{11} = 10\frac{10}{11} = 10.909090...$

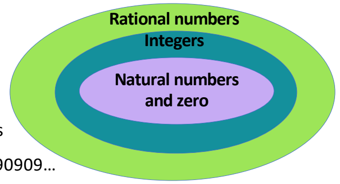
We need to extend our ideas of arithmetic

to include fractions (rational numbers).

Then all such equations have solutions.

Notice the answer is a rational number which has

an infinite decimal with the recurring pattern 9090909…



24

A man owned 19 camels and he died.
Before his death he made a will
that half should go to his son,
a quarter to his daughter and
one fifth to his servant

How many camels would each person get?
The family could not agree how to share the camels.
Can you help them?
Who would be happy with this sharing:
Son: $9\frac{1}{2}$ Daughter: $5\frac{1}{2}$ Servant: 4 ? Why?

Son: $9\frac{1}{2} = \frac{19}{2}$ Daughter: $5\frac{1}{2} = \frac{22}{4} > \frac{19}{4}$ Servant: $4 = \frac{20}{5} > \frac{19}{5}$ Camel for dinner. Is everyone happy?

IS THIS A BETTER SOLUTION? WHAT IS WRONG WITH IT?

A wise man who was travelling on his camel came to their rescue and solved the problem as follows:
Along with the 19 camels he also included his own camel to make 20 camels.

Son: $\frac{1}{2}$ of 20 camels = 10 Daughter: $\frac{1}{4}$ of 20 camels = 5 Servant: $\frac{1}{5}$ of 20 camels = 4

A total 10 + 5 + 4 = 19 camels. The wise man took his camel and rode away.

25

WHAT IS THE ADDITIVE IDENTITY?
DOES EVERY RATIONAL NUMBER HAVE AN ADDITIVE INVERSE?

Odds, evens and rational numbers

Addition and subtraction are **inverse operations.**

Complete these calculations

$^{+}5 + ^{-}5 =$ **0**

$^{+}n + ^{-}n =$ **0** for all n

Zero is the **identity for addition.**

Numbers ^{+}n and ^{-}n are **additive inverses.**

Subtraction is like adding an inverse.

26

WHAT IS THE IDENTITY FOR MULTIPLICATION?
DOES EVERY RATIONAL NUMBER HAVE A MULTIPLICATIVE INVERSE?

Just as addition and subtraction are **inverse operations, multiplication and division are also inverse operations.**

Complete these calculations:

5 × 4 = 20

20 ÷ 4 = [ ] **$20 \div 4 = 20 \times \frac{1}{4} = 5$**

4 × 1 = **4**

$n \times 1 =$ **n**

For all numbers n, dividing is the same as multiplying by an inverse.

$4 \times \frac{1}{4} =$ **1**

$n \times \frac{1}{n} =$ **1**

1 is the **identity for multiplication.**

Every rational number n (except 0) has multiplicative inverses,

$\frac{1}{n}$ and n.

27

WHAT IS ASSOCIATIVITY?

ADDITION AND SUBTRACTION

(7 + 5) + 2 =
7 + (5 + 2) =
Are the answers the same?

(7 + 5) + 2 = 14
7 + (5 + 2) = 14
Yes, the answers are the same.
Addition is **associative.**

(7 – 5) – 2 =
7 – (5 – 2) =
Are these answers the same?

(7 – 5) – 2 = 0
7 – (5 – 2) = 4
No, these answers are different.
Subtraction is **not associative**

MULTIPLICATION AND DIVISION

(7 × 5) × 2 =
7 × (5 × 2) =
Are the answers the same?

(7 × 5) × 2 = 70
7 × (5 × 2) = 70
Yes, the answers are the same. Multiplication is **associative.**

(7 ÷ 5) ÷ 2 =
7 ÷ (5 ÷ 2) =
Are these answers the same?

$(7 \div 5) \div 2 = \frac{7}{10}$
$7 \div (5 \div 2) = \frac{14}{5}$
No these answers are different. Division is **not associative**

We use brackets to make the order of operations clear.

28

WHAT IS COMMUTATIVITY?

5 + 7 = 7 + 5 = 12 — Yes, so we say addition is **commutative.**

but is 5 – 7 the same as 7 – 5? — No, subtraction is **not commutative**.

SIMILARLY

5 × 7 = 7 × 5 = 35 — Yes, multiplication is **commutative.**

but is 5 ÷ 7 the same as 7 ÷ 5? — No, division is **not commutative.**

29

SUMMARY OF THE RULES FOR THE ADDITION OF RATIONAL NUMBERS

Remember subtracting is just adding an inverse.

RULES FOR CALCULATING

The rules of arithmetic that are taught in school are simple.

All the number work done in school follows the following rules:

(1) ADDITION

- The set of rational numbers is ***closed under addition***,
- For all *a, b* and *c*, $(a + b) + c = a + (b + c)$ (***associative***)
- The rational number zero is the ***additive identity***
- Every rational number has an ***additive inverse***.

This is an example of ***a mathematical system called a GROUP .***

30

SUMMARY OF THE RULES FOR MULTIPLICATION OF RATIONAL NUMBERS

Remember dividing is just multiplying by an inverse.

RULES FOR CALCULATING

(2) MULTIPLICATION

- The set of rational numbers, leaving out the number zero, is ***closed under multiplication***.
- For all a, b and c, $(a \times b) \times c = a \times (b \times c)$ (***associative***).
- The rational number 1 is the ***multiplicative identity.***
- Every rational number in this set has a ***multiplicative inverse***.

Addition and multiplication are ***commutative*** : $a + b = b + a$

and $a \times b = b \times a$ for all a and b.

Rational numbers, leaving out the number zero, **form another group**.
We now have rules for + and x. Now let's put everything together.

31

SUMMARY OF THE RULES FOR +, -, × AND ÷ OF RATIONAL NUMBERS

(3) THE DISTRIBUTIVE LAW FOR COMBINING ADDITION AND MULTIPLICATION

When we add and multiply rational numbers we use the ***distributive law***.

For example 3 × (4 + 5) =

(3 × 4) + (3 × 5) = 27.

	4	5
3	12	15

RULES FOR CALCULATING

Rules (1), (2) and (3) describe additive and multiplicative **groups** for rational numbers and the distributive law.
The two groups together with the distributive law make up **what mathematicians call a FIELD.**

32

FRACTIONS (or RATIONAL NUMBERS)

What decimal do you get for the number $\frac{4}{33}$?
Use your calculator: is there a pattern?

$\frac{4}{33}$ = 0.121212... This goes on for ever repeating 12 again and again.

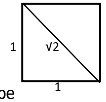
Choose your own fraction and then divide the numerator by the denominator.

Do you always find a pattern in the decimal digits?

Sometimes the decimal is recurring like $\frac{4}{33}$.
Sometimes it is like $\frac{3}{4} = 0.7500$... when the recurring 0 can be ignored and the decimal terminates.

Is your answer recurring or does it terminate?
Try another fraction.

IRRATIONAL NUMBERS



The length of the diagonal of a unit square is √2.
This is an **irrational number**, it cannot be written as a fraction.
Also π is an irrational number and it can't be written as a fraction.
Approximately, √2 = 1.414 and π = 3.142
There is no pattern to the digits after the decimal point. YOU WILL NEVER GET THESE NUMBERS EXACTLY HOWEVER HARD YOU TRY.
What does your calculator give for √2?
Is there a pattern?
What about π ? Is there a pattern?

√2 = 0.14142135623... and it goes on and on without a pattern!
π = 3.1415926535... and it goes on and on without a pattern!

33

IRRATIONAL NUMBERS

Rational numbers and irrational numbers together make up the **real numbers**.

We have already discussed the field of rational numbers.

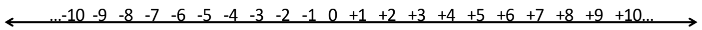
The real numbers obey the same rules for addition and multiplication as the rationals.

So, there is a field of real numbers.

34

DO WE NEED TO GO BEYOND REAL NUMBERS?

The rational and irrational numbers together make up the ***real numbers***. Each real number corresponds to exactly one point on a line and all the points on that line are represented by real numbers.
We call this line *the* ***real line*** .



What is the solution to $x^2 = 2$? **$x = {}^{+}\sqrt{2}$ and $x = {}^{-}\sqrt{2}$ are solutions to $x^2 = 2$.**

What are the solutions to the equation $x^2 = -2$?

If we only know about real numbers, we cannot solve the equation $x^2 = a$ when a is negative because squares of real numbers are always positive or zero.

Using the formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ for solutions of the quadratic equation $ax^2 + bx + c = 0$, solve the equations (1) $x^2 - 2x - 7 = 0$ and (2) $x^2 + x + 1 = 0$. Are the solutions rational? Are they real?

(1) $x = \frac{1}{2}(2\pm\sqrt{32}) = 1 \pm 2\sqrt{2}$ which are irrational and real.

(2) $x = \frac{-1\pm\sqrt{-3}}{2}$ which are not real because $\sqrt{-3}$ is not real.

35

ONE DIMENSIONAL REAL NUMBERS
TWO DIMENSIONAL COMPLEX NUMBERS

Real numbers are good enough for many mathematical purposes and are used in industry, commerce, science and engineering.

But, as we have seen with quadratic equations, they are not good enough for all purposes.

We need to use **two dimensional numbers, the complex numbers**.

With complex numbers you can solve **ALL** quadratic equations.

36

TWO DIMENSIONAL NUMBERS

Whereas real numbers correspond to points on a line, complex numbers correspond to points in the plane, so complex numbers are two dimensional.

The complex number written as $x + yi$ corresponds to the point in the plane with coordinates (x , y).

Let's find out more about the mysterious $\boldsymbol{i}$ which is called an 'imaginary' number and which corresponds to the point (0 , 1).

$\boldsymbol{i} = \sqrt{-1}$ and it it is important because we can use it to solve equations.

37

ADDING, SUBTRACTING, MULYIPLYING AND DIVIDING COMPLEX NUMBERS

Just using the algebra you know, work out the following examples:

$(7 + 6i) + (2 + 4i) = 9 + 10i$

$(9 + 10i) - (2 + 4i) = 7 + 6i$

$(7 + 6i) \times (2 + 4i) = 14 + 24i^2 + 28i + 12i$ remember $i = \sqrt{-1}$

$= 14 - 24 + 40i$ and $i^2 = -1$

$= -10 + 40i$

Hint: use this multiplication to find the next answer.

$(-10 + 40i) \div (2 + 4i) = 7 + 6i$

You know the rules for +, -, x, ÷ real numbers.
The same rules hold for addition, subtraction, multiplication and division of complex numbers $x + yi$ where $i^2 = -1$.

In mathematical language, the **rational numbers**, the **real numbers** and the **complex numbers** are **three fields of numbers with the same rules**.

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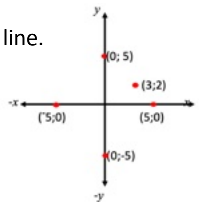
THE 2-DIMENSIONAL COMPLEX NUMBERS AND THE COMPLEX PLANE



Every real number corresponds to a point on the real number line.

Every complex number corresponds to a point in the plane.
The point (x , y) is the complex number $x + iy$

The point in the plane with coordinates (5 , 0) corresponds to the complex number written as $5 + (0 \times i)$ (the real number 5 on the real line).

The point in the plane with coordinates (3 , 2) corresponds to complex number written as $3 + 2i$

The point (0 , 5) corresponds to $0 + 5i$

The point (-5 , 0) corresponds to -5 (it's also a real number on the real line.)

The point (0 , -5) corresponds to $0 - 5i$

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ALL POLYNOMIAL EQUATIONS OF DEGREE *n* HAVE *n* SOLUTIONS

1777 – 1823

Carl Friederich Gauss

In 1799 Gauss proved the Fundamental Theorem of Algebra that every polynomial equation has a full set of complex solutions, that is the same number of solutions as the highest power in the polynomial.

(Note that the solutions may be repeated when the graph of the polynomial touches the x-axis giving a repeated solution at that point).

This means that, with complex numbers:

- every quadratic equation has two solutions,
- every cubic has three solutions, and so on.
- For example the quartic $x^4 - 5x^3 + x^2 - 7x + 12 = 0$ is a polynomial equation with **4** solutions because **4** is the highest power of x.

Finally, about 200 years ago complex numbers were accepted as 'proper' numbers.

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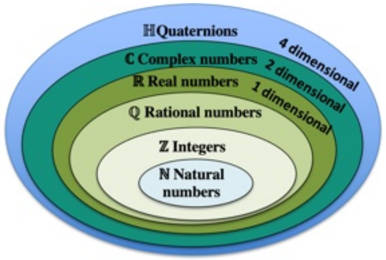
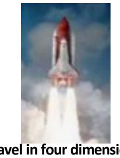
One dimensional real numbers can be generalised to two dimensional complex numbers. What about higher dimensional numbers?

Travel in one dimension: train on a straight track.

Travel in three dimensions: aeroplane on flightpath.



Travel in two dimensions: ship on the sea.



Travel in four dimensions: rocket in space space-time.

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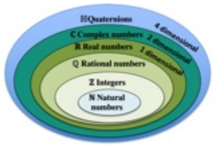
FOUR DIMENSIONAL NUMBERS - QUATERNIONS

Quaternions, discovered by the Irish mathematician Sir William Rowan Hamilton in 1843, are useful **4-dimensional numbers**.

Complex numbers are just pairs of real numbers and quaternions are pairs of complex numbers.

William Hamilton
1805 - 1865



Quaternions are an important tool frequently used in programming computer graphics and controlling machines, because the equations of rotations and reflections in 3 dimensions are given in a very simple form by quaternions.

Applied mathematics and physics deal with motions in space so quaternions are very useful there, for example in Einstein's space-time equations.

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THE FIRST PICTURES FROM THE JAMES WEBB TELESCOPE

The red smudge is 35 **BILLION** light-years away.
We see it, as it was, just 235 million years after the Big Bang.

The James Webb Space Telescope, launched in December 2021, is a $10 billion machine in search of the edge of darkness and the beginning of time.
It is an infrared observatory orbiting the Sun about 1 million miles from Earth to find the first galaxies that formed in the early universe and to see stars forming planetary systems.

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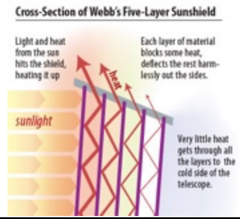
NASA employees stand by a full-scale replica of the James Webb Telescope.



The telescope folded to pack into the rocket.

https://www.nasa.gov/mission_pages/webb

https://webbtelescope.org/news/first-images/gallery

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James Webb Space Telescope

The telescope had to be unfolded and adjusted so that the sunshield protects the instruments from the Sun's rays.

The number knowledge we have thought about today was used to calculate the exact angles to control the telescope and distances less than a hundredth of a millimetre to 35 billion light-years.

Look at the yellow primary mirror. It is made up of separate hexagons that fit together. The mirrors capture the light rays that make the pictures from outer space. Each mirror is individually adjusted for the pictures to be in focus.

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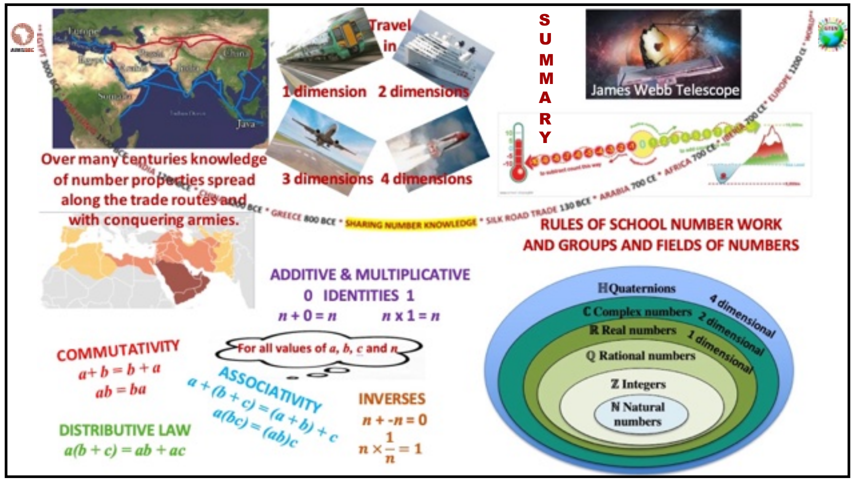

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NUMBERS IN THE LAST 5000 YEARS AND NUMBERS IN YOUR LIVES

Today we have met a 'big picture', without getting too technical, ranging from kindergarten mathematics (counting numbers) to the fringe of research (quaternions).

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News about the James Webb Telescope **https://www.nasa.gov/mission_pages/webb/main/index.html**

NRICH Articles:	What are Numbers?	https://nrich.maths.org/5805
	Small Groups	https://nrich.maths.org/1328/
	An Introduction to Mathematical Structure	https://nrich.maths.org/2769/
	What Are Complex Numbers?	https://nrich.maths.org/2432/
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	Ubiquitous Octonions	https://plus.maths.org/content/ubiquitous-octonions
Other websites:	History of Numbers and Number Theory	https://mathshistory.st-andrews.ac.uk/HistTopics/category-number-theory/
	Wikipedia on Quaternions	https://en.wikipedia.org/wiki/Quaternion
	Wikipedia on Clifford Algebras	https://en.wikipedia.org/wiki/Clifford_algebra

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LET'S PLAY MATHEMATICALLY AND LEARN

Order from AMAZON or TARQUIN https://www.tarquingroup.com/products/aiming-high-lets-play-mathematically

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- to unlock knowledge
- to improve numeracy and visualisation skills
- to practise mathematical procedures
- to motivate concentration and critical thinking
- to boost confidence in mathematical ability.

This first book in the AIMING HIGH series provides 36 games that are easy to learn and enjoyable to play for any age. Each comes with reflective questions and materials designed to bring out mathematical thinking and provide a deeper understanding of the topic that underlies the game. Even for the youngest players, this can be transformational.

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Thanks for coming to this workshop.

Use the AIMSSEC ideas
on AIMING HIGH and add comments.

Share what you have learned
with other teachers.

Try to help all your learners to have a
'YES I CAN'
attitude to mathematics.

Toni Beardon LAB11@cam.ac.uk
Caroline Ainslie caroline@bubblymaths.co.uk

Enquire about signing up for an AIMSSEC course
as a self-funding student admin@aimssec.ac.za

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