Global Teacher Empowerment Network GTEN
THE GOLDEN RATIO IN MATHS, SCIENCE AND ART
Saturday 13 April 2024 16:00 – 18:00 London Time


Toni Beardon


Caroline Ainslie


Sam Okoth

1

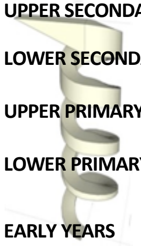
AIMS African Institute for Mathematical Sciences SCHOOLS ENRICHMENT CENTRE

MATHS TOYS

Global Teacher Empowerment Network (GTEN)
PROGRAMME: THE GOLDEN RATIO IN MATHEMATICS SCIENCE AND ART

IMPROVE SKILLS, KNOWLEDGE AND UNDERSTANDING OF:
- Number patterns and sequences
- Ratios
- Spirals
- Geometric constructions
- Similar and congruent triangles
- Properties of a regular pentagon
- Quadratic Equations
- Connections between mathematics, art, architecture and science.

LEARNING SPIRAL



UPPER SECONDARY
- 12. The Golden Ratio by square roots and by continued fractions.
- 11. Euler's definition of the Golden Ratio.

LOWER SECONDARY
- 10. Solving the quadratic equation that gives the Golden Ratio.
- 9. Construction of the Golden Rectangle by ruler and compass and by paper folding.

UPPER PRIMARY
- 8. The Golden Ratio in a Regular Pentagon.
- 7. Examples of the Golden Ratio in Art, Architecture and Science

LOWER PRIMARY
- 5. The ratios of successive terms in the Fibonacci Sequence
- 4. The Elephant Dreaming spiral
- 2. The Sheep Talk number sequence

EARLY YEARS
- 1. Starter activity: tie a knot in a strip of paper making a pentagon

2

During this session you need to wear 2 hats to appreciate how a learner at a particular stage would do the activity and what they would learn from it.

Teacher

Learner

AIMSSEC

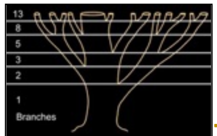
DO-TALK-RECORD

is the basis of a good lesson

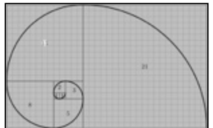
DO the activities today pretending that you only know what your learners know.

TALK to other teachers. How might learners discover new ideas by doing these activities?

RECORD for yourself as a teacher and also in a way that would help learners.



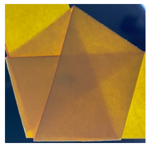
DO Count branches at each level.
TALK What do you notice?
RECORD Make notes.


DO Study this spiral.
TALK What do you notice?
RECORD Copy the spiral accurately.

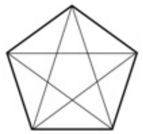
3

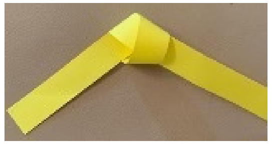
The Pentagram and The Golden Ratio ϕ

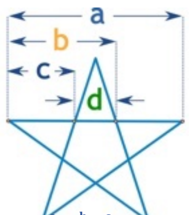
AIMSSEC



DO: Take a strip of paper and tie a knot, then flatten it carefully. Hold it up to the light. Can you see the pentagon with the five pointed star or pentagram inside?





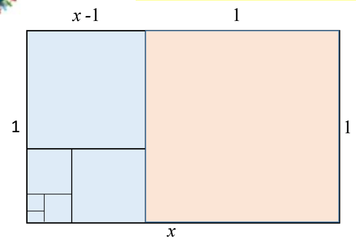


$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \emptyset$

The pentagram star symbol appears in many cultures and in some of the flags of the world.

4

The Golden Rectangle and The Golden Ratio ϕ



The Golden Rectangle has the property that it can be divided into a rectangle and a square where **the inner rectangle has the same proportions as the outer rectangle.**

The ratio of the long to short edges is the same.

This process of splitting the rectangle into a square and a smaller rectangle of the same proportions can be repeated again and again indefinitely.

The ratios of the long to short edges of the Golden Rectangle are $\frac{x}{1}$ and $\frac{1}{x-1}$

The Golden Ratio is given by solving the equation $\frac{x}{1} = \frac{1}{x-1}$

The Golden Ratio has been observed and studied for thousands of years. It has influenced designs in architecture and art.

5

SHEEP TALK

Sheep Talk The first sheep talk word is **A**. To find the next term in the word sequence, each A is replaced by B and each B is replaced by AB.
The rule is: **A -> B and B-> AB**

Here are the first words in the sheep talk sequence: A B AB BAB ABBAB BABABBAB ABBABBABABBAB

DO **Continue the sequence of words and count the number of occurrences of A, of B, and the total number of letters.**

RECORD **Fill in the table. Spot the patterns in the sequences and**
TALK **about why the patterns occur.**

Number of A's										
Number of B's										
Number of letters										

6

SHEEP TALK

Sheep Talk The rule is: **A -> B and B-> AB**
Here are the first 7 words in the sheep talk sequence:

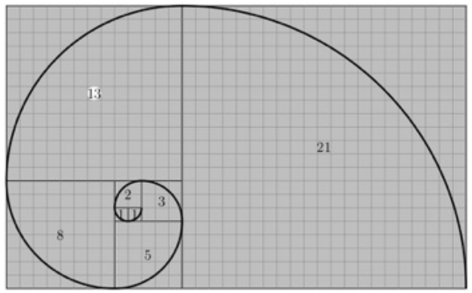
A B AB BAB ABBAB BABABBAB ABBABBABABBAB BABABBABABBABBABABBAB

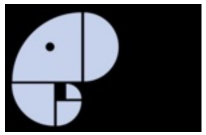
Number of A's	1	0	1	1	2	3	5	8	13	21	34
Number of B's	0	1	1	2	3	5	8	13	21	34	55
Number of letters	1	1	2	3	5	8	13	21	34	55	89

7

ELEPHANT DREAMING FIBONACCI SPIRAL

DO: Copy the picture below on squared paper, draw the spiral and create your own elephant.





The numbers in the centres of the squares give the radii of the quarter circles you need to draw. For the elephant's trunk, start from the quarter circle of radius 3 and don't draw the quarter circles with radii 1, 1, and 2.

8

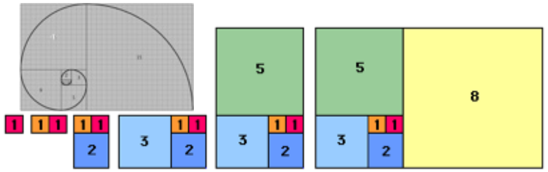
The Fibonacci Sequence

The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

What is the rule for finding the next term in the sequence?

The ratios of the consecutive terms 1, 2/1, 3/2, 5/3, 8/5, 13/8 …
are the ratios of the edge lengths of the rectangles (long edge to short edge)

Write these fractions as decimals and continue the sequences.
What do you notice?



FIBONACCI SQUARES

Fibonacci number	Ratio of consecutive terms
1	
1	1
2	2
3	1.5
5	1.6666667
8	1.6
13	1.625
21	1.6153846
34	1.6190476
55	1.6176471
89	1.6181818
144	1.6179775
233	1.6180556
377	1.6180258
610	1.6180371
987	1.6180328
1597	1.6180344
2584	1.6180338
4181	1.6180341
6765	1.618034
10946	1.618034
17711	1.618034
28657	1.618034
46368	1.618034

9

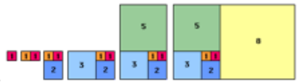
The Fibonacci Sequence and The Golden Ratio

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

The ratios of successive terms are:

1, 2, 1.5, $1.\overline{6}$, 1.6, 1.625, 1.615, 1.619, 1.618…

These ratios converge to The Golden Ratio ϕ.



FIBONACCI SQUARES

Did you notice that adding two successive terms in the Fibonacci sequence gives the next term in the sequence
1 + 1 = 2, 1 +2 = 3, 2 + 3 = 5 etc. ?

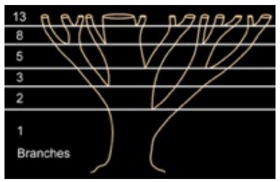
The sequence is defined, giving the n^{th} term as F_n, by the formula:

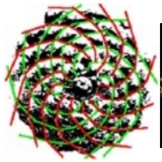
$F_n = F_{n-1} + F_{n-2}$ with first two terms $F_0=1$ and $F_1=1$

10

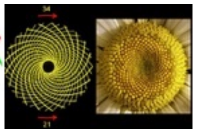
WHAT TO YOU NOTICE?

Lily Euphorbia Trilium Columbine Daisy





Pinecone

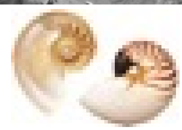


Sunflower

Nautilus shell

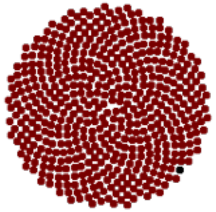
11

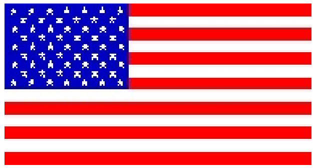
The Golden Ratio appears in Art, Architecture and Nature



12

https://www.geogebra.org/m/B4C9bbuy





13

GEODESIC DOMES

Bucky ball

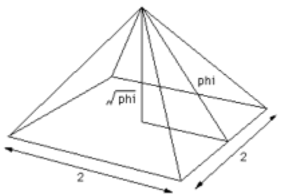
Eden Project Biome Greenhouses

Regular pentagons and hexagons form the Buckminster fullerene or bucky ball, as in soccer balls. Geodesic Domes can be built with the same configuration using irregular polygons.

Geodesic Domes are energy efficient. Compared to other structures, light reflects and stays in the dome longer, the high volume to surface area ratio means less materials are used in construction, and heating costs are low.

14

The Great Golden Pyramid of Giza c. 2560 BC



St Peter's, Rome

St Paul's, London

Westminster Cathedral

Milan Cathedral

Florence Cathedral

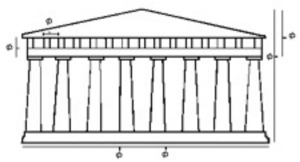
The structure consists of approximately 2 million blocks of stone, weighing 6 million tonnes in total.

The Great Pyramid, with dimensions: 230 m by 230 m and height 146.6 m, is bigger than any of the great cathedrals. For example it's bigger than Milan Cathedral which holds 40,000 people.

Man fears Time,
yet Time fears the Pyramids
Arab proverb

15

The Golden Ratio (or Divine Proportion) in Art and Nature

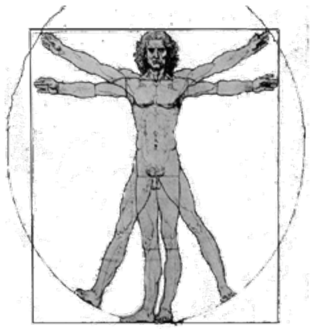


The Parthenon in Athens, built in 447 BC, is an example of the use of the golden ratio in architecture.

http://academic.reed.edu/humanities/110Tech/Parthenon.html

16

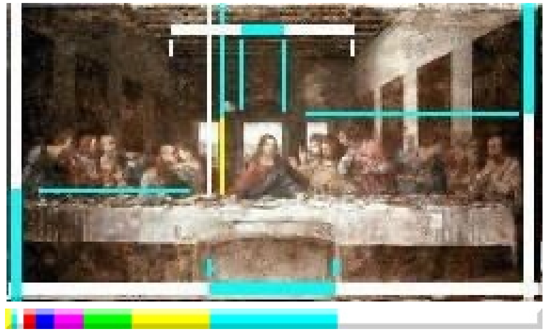
The Divine Proportion in Art and Nature



In this drawing by Leonardo da Vinci the height of a person is divided into two sections, the dividing point being the navel. The distance from the navel to the the soles of the feet divided by the distance from the top of the head to the navel was equal to 1.618, the golden ratio.

17

The Last Supper Leonardo da Vinci 1452-1519 Convent of Santa Maria delle Grazie, Milan



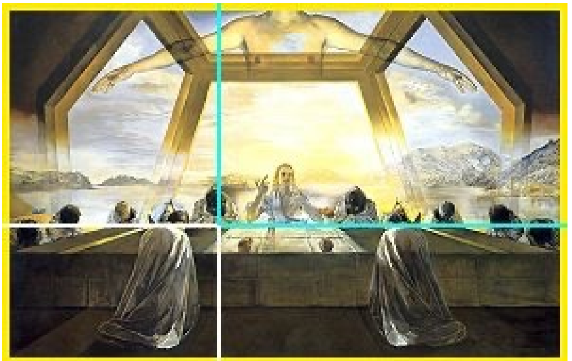
The Golden Ruler

The Golden Ruler, and the markings on the painting, show the two, symmetrically placed, positions of the golden section points.

The coloured sections on the golden ruler show the section points as the rectangles are split into squares and rectangles at smaller and smaller scales.

18

Salvador Dali The Last Supper 1955



Salvador **Dali** 1904-1989 "The Sacrament of the **Last Supper**". National Gallery of Art, Washington DC. 1955

The painting shows part of a dodecahedron. The golden ratio occurs in many ways in regular pentagons and regular dodecahedra.

19

The Golden Ratio is also called The Divine Proportion or Golden Mean or Golden Section or Golden Number

Leonardo of Pisa (Fibonacci) was the son of a merchant and travelled widely, meeting with many merchants and learning about their methods of doing arithmetic. He realised the many advantages of the Hindu-Arabic number system, which, unlike the Roman numerals used at the time, allowed easy calculation using a place-value system.

His book Liber Abaci (*Book of Abacus* or *The Book of Calculation*), popularized Hindu–Arabic numerals and revolutionised calculation methods in Europe.

In this book Leonardo also wrote about the the Golden Ratio and the sequence which is named after him.

https://en.wikipedia.org/wiki/Fibonacci

20

Fibonacci's Rabbits Problem

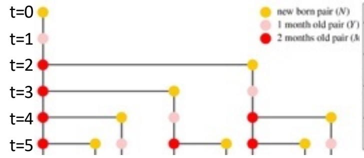
Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Young rabbits mate at the age of one month, so that at the end of its second month a female is mature and can produce another pair of rabbits.

Suppose that our rabbits have long lives and that the female always produces one new pair (one male, one female) every month from the second month on.

How many rabbits will there be at the end of 3 months?
How many at the end of one year?

21

Fibonacci's Rabbits



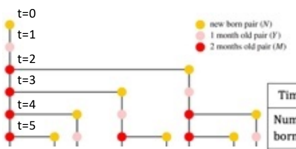
Time in months	0	1	2	3	4	5	6	7	8	9	10	11	12
Number of new-born pairs N_t													
Number of young pairs Y_t													
Number of mature pairs M_t													
Total number of pairs F_t													

The diagram shows the population of rabbits at times t = 0, 1, 2, 3, 4 and 5 months.
At time t = 0 the original pair of new-born rabbits are shown by a yellow disc.
At time t = 1 month the same pair are shown in pink, as young rabbits.
At time t = 2 months the diagram shows the same rabbits (now mature) in red together with a pair of new-borns (in yellow).
At time t = 3 months the original pair have produced a 2nd pair of new-borns (orange) and their 1st offspring are now a young pair and shown in pink.
The population is also shown at times t = 4 and 5 months.
Extend the tree diagram to show the population at t = 6 months and fill in the table.

22

Fibonacci's Rabbits

Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits mate at the age of one month, so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits have long lives, and that the female always produces one new pair (one male, one female) every month from the second month on.

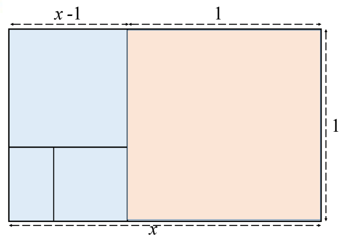


Time in months	0	1	2	3	4	5	6	7	8	9	10	11	12
Number of new-born pairs N_t	1	0	1	1	2	3	5	8	13	21	34	55	89
Number of young pairs Y_t	0	1	0	1	1	2	3	5	8	13	21	34	55
Number of mature pairs M_t	0	0	1	1	2	3	5	8	13	21	34	55	89
Total number of pairs F_t	1	1	2	3	5	8	13	21	34	55	89	144	233

What do you notice?

23

The Golden Rectangle and The Golden Ratio ϕ



The ratios are: $\frac{x}{1} = \frac{1}{x-1}$

This leads to the equation $x^2 - x - 1 = 0$

Solve this equation.

The solutions are: $(1 \pm \sqrt{5})/2$

The positive solution $x = (1 + \sqrt{5})/2$

is the Golden Ratio which is denoted by

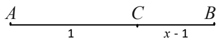
the Greek letter ∅ = 1.618 to 4 s.f.

24

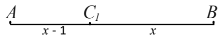
Golden Ratio ϕ

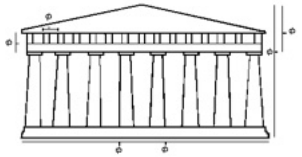
Euclid in *The Elements*, says that the point C divides the line *AB* in the Golden Ratio if and only if the extreme ratio AB : AC is equal to the mean ratio AC : CB

AB : AC = AC : CB



Similarly $BA:BC_l=BC_l:C_lA$





If $AB = x$ units and $AC = 1$ unit

then $AB : AC = AC : CB$

becomes $x : 1 = 1 : x - 1$

or $\frac{x}{1} = \frac{1}{x-1}$ Do you recognize this equation?

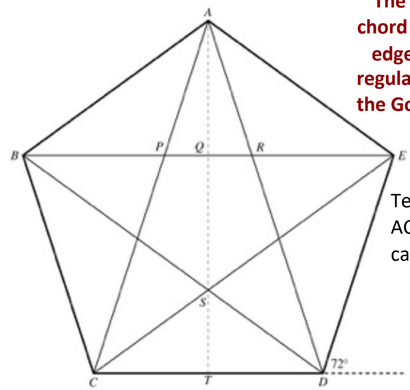
This leads to the equation $x^2 - x - 1 = 0$.

We have solved this equation. The solutions are: $(1 \pm \sqrt{5})/2$

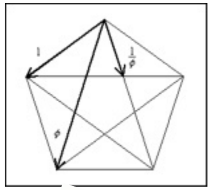
The positive solution $x = (1 + \sqrt{5})/2$ is called the Golden Ratio denoted by the Greek letter ∅

25

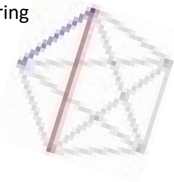
The Pentagram and The Golden Ratio ϕ



The ratio of the chord length to the edge length in a regular pentagon is the Golden Ratio ϕ



Test this by measuring AC and AB and calculating $\frac{AC}{AB}$

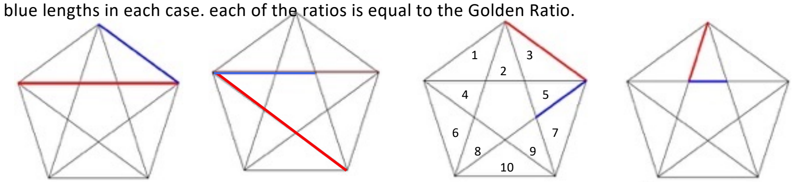


26

The Regular Pentagon and The Golden Ratio ϕ

https://aiminghigh.aimssec.ac.za/golden-pentagon/

Measure the red and blue lengths in each of these pictures and calculate the ratio of the red to the blue lengths in each case. each of the ratios is equal to the Golden Ratio.

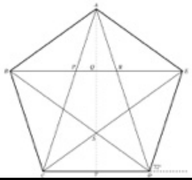


Proofs of these results involve similar triangles.

List triangles 1 to 10 grouping them into sets of congruent triangles

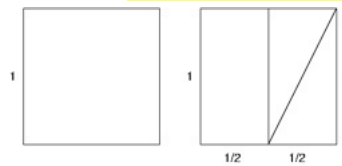
Can you spot all the similar triangles?

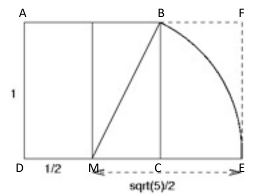
Can you work out all the angles?



27

A construction for the Golden Rectangle





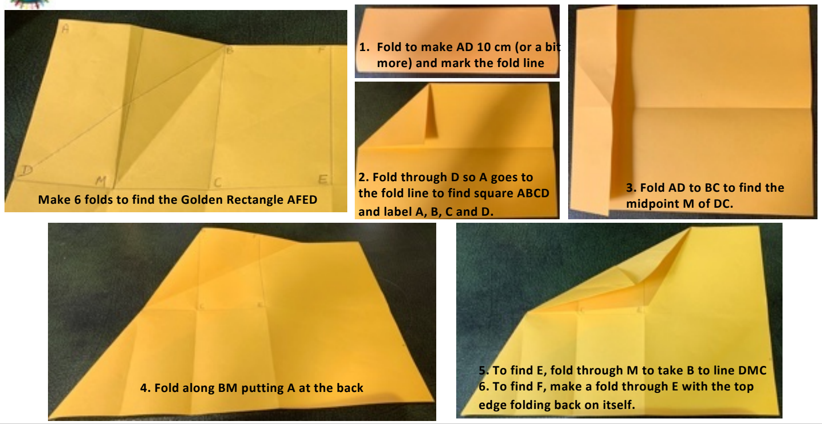
1. Draw a square of side 1 unit near the left side of the paper (10 cm is a convenient length)
2. Bisect the base
3. Draw the diagonal to form a triangle with sides 1 unit and 1/2 unit so that the diagonal has length $\sqrt{5}/2$
4. Draw the arc BE of the circle centre M with radius MB and find the point E where this arc cuts the extended base to give one vertex of the golden rectangle AFED.

As ME $=\sqrt{5}/2$ the edges of AFED are $(1+\sqrt{5})/2$ and 1 (ratio ϕ:1) so AFED is a Golden Rectangle.

If you have no compasses you can measure the distance MB but it won't be as accurate.

28

The Golden Rectangle by paper-folding



29

$$x=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$$

What do you notice in this formula?

$$x^2 = 1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$$
$$= 1+x$$

$x^2 - x - 1 = 0$ We have already solved this equation.

The solutions are: $(1 \pm \sqrt{5})/2$ $= (1+\sqrt{5})/2 = 1.618$ to 4 s.f.

The value of x given by this formula is equal to the Golden Ratio

$$\phi=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$$

30

Continued Fractions

What do you notice about this sequence? $x,\quad 1+\frac{1}{x},\quad 1+\frac{1}{1+\frac{1}{x}},\quad 1+\frac{1}{1+\frac{1}{1+\frac{1}{x}}},\ldots$

Again, a repeating pattern where x is replaced by $1 + 1/x$.
The pattern goes on and on for ever.

Suppose we choose a value of x and calculate the values of these fractions.
Let's do this for $x = 2$

What do you notice?

$$1+\frac{1}{2}=\frac{3}{2};$$
$$1+\frac{1}{1+\frac{1}{2}}=1+\frac{1}{3/2}=1+\frac{2}{3}=\frac{5}{3};$$
$$1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}=1+\frac{1}{5/3}=1+\frac{3}{5}=\frac{8}{5};$$
$$1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}}=1+\frac{1}{8/5}=\frac{13}{8}$$

31

Continued Fractions

$x,\quad 1+\frac{1}{x},\quad 1+\frac{1}{1+\frac{1}{x}},\quad 1+\frac{1}{1+\frac{1}{1+\frac{1}{x}}},\ldots$

We calculate the values of these fractions for $x = 2$
Think of each continued fraction as $1+\frac{1}{x}$ and look at the colours. You will see that the denominator x corresponds to the continued fraction BEFORE it in the sequence.

For $x = 2$ the sequence is:

$$2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots$$

The continued fraction values give the sequence of ratios of the Fibonacci numbers that we saw earlier which converges to $\emptyset$ **= 1.618 to 4 sf.**

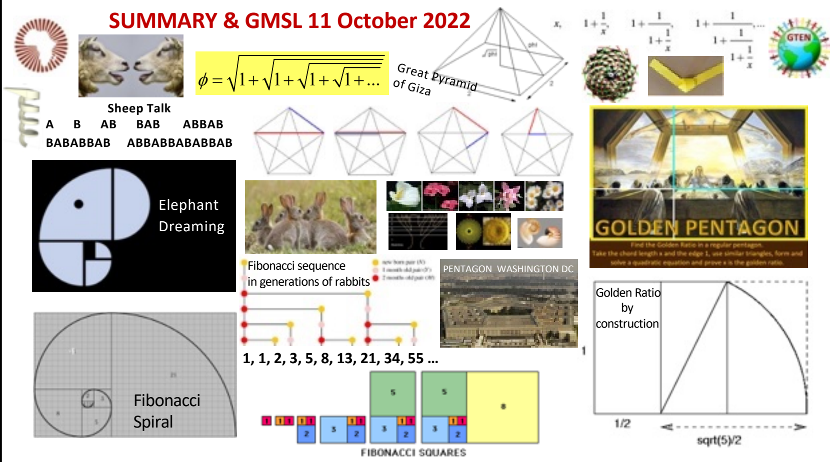
$x = 2$: $1+\frac{1}{2}=\frac{3}{2};$

$x = 3/2$: $1+\frac{1}{1+\frac{1}{2}}=1+\frac{1}{3/2}=1+\frac{2}{3}=\frac{5}{3};$

$x = 5/3$: $1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}=1+\frac{1}{5/3}=1+\frac{3}{5}=\frac{8}{5};$

$x = 8/5$: $1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}}=1+\frac{1}{8/5}=\frac{13}{8}$

32



33

AIMINGHIGH TEACHER NETWORK

AIMSSEC

On the AIMING HIGH website there are freely downloadable worksheets, Notes for Teachers with solutions, Key Questions and Diagnostic Quizzes for formative assessment, also Inclusion and Home Learning Guides with activities for learners of all ages and attainment levels.

SHEEP TALK https://aiminghigh.aimssec.ac.za/sheep-talk/
ELEPHANT DREAMING https://aiminghigh.aimssec.ac.za/elephant-dreaming/
GOLDEN PENTAGON https://aiminghigh.aimssec.ac.za/golden-pentagon/
ONE STEP TWO STEPS https://aiminghigh.aimssec.ac.za/one-step-two-steps/
FIBONACCI'S RABBITS https://aiminghigh.aimssec.ac.za/fibonaccis-rabbits/
NATURE BY NUMBERS Cristobal Vila https://youtu.be/kkGeOWYOFoA/
A SUMMARY FOR THE END OF A SEQUENCE OF LESSONS Ben Sparks https://youtu.be/z9d1mxgZ0ag

34

LET'S PLAY MATHEMATICALLY AND LEARN

Order from AMAZON or TARQUIN https://www.tarquingroup.com/products/aiming-high-lets-play-mathematically

Play Mathematically
- to develop a love for mathematics
- to unlock knowledge
- to improve numeracy and visualisation skills
- to practise mathematical procedures
- to motivate concentration and critical thinking
- to boost confidence in mathematical ability.

This first book in the AIMING HIGH series provides 36 games that are easy to learn and enjoyable to play for any age. Each comes with reflective questions and materials designed to bring out mathematical thinking and provide a deeper understanding of the topic that underlies the game. Even for the youngest players, this can be transformational.

35

AIMS African Institute for Mathematical Sciences SCHOOLS ENRICHMENT CENTRE

Thanks for coming to this workshop.

Use the AIMSSEC ideas on AIMING HIGH and add comments.

Share what you have learned with other teachers.

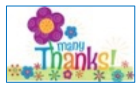
Try to help all your learners to have a

'YES I CAN'

attitude to mathematics.

Toni Beardon LAB11@cam.ac.uk
Caroline Ainslie caroline@bubblymaths.co.uk

Enquire about signing up for an AIMSSEC course as a self-funding student admin@aimssec.ac.za



36