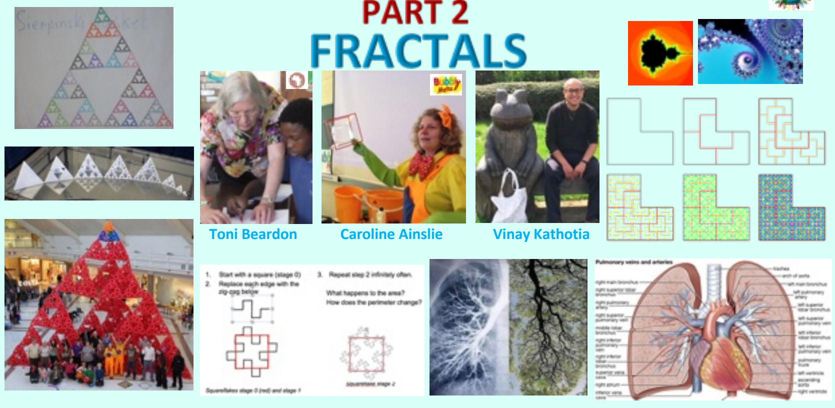
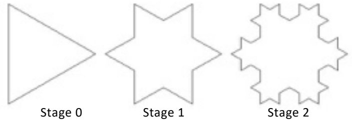
Global Teacher Empowerment Network GTEN

PART 2
FRACTALS



28

VON KOCH CURVE



Every line segment is replaced by a zig-zag curve made up of 4 pieces

at each stage of the repeated process forming the fractal.

The Von Koch curve gets longer and longer like the squareflake curve. The area does not stay the same but it only increases a little.

Making a poster is a good class activity. The protruding V shapes can be added to the poster by drawing or sticking on smaller triangles.

See follow-up slides 50 - 56

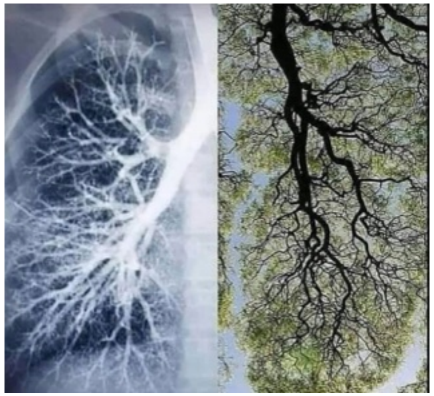
29

FROM 2D TO 3D - FRACTALS IN NATURE

Evolutionary development has produced these fractal forms to optimise the exchange of gases by maximising the surface area within a given volume.

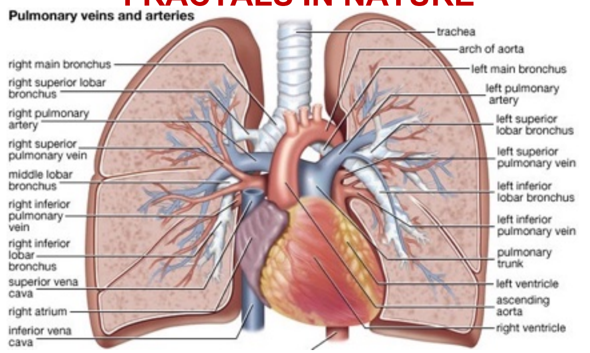
Trees absorb carbon dioxide and emit oxygen so they are vital to the environment.

People breathe in oxygen which they need for life, and they breathe out carbon dioxide.



30

FRACTALS IN NATURE



3 fractal systems: veins, arteries and lungs.
Limited volume but very large surface area

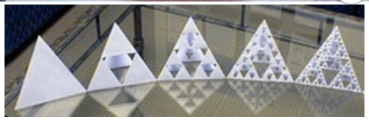
31

SIERPINSKI TETRAHEDRON FROM 2 DIMENSIONS TO 3

Balloon Model for Cambridge Science Week March 2014 Cambridge Grafton Centre

To raise money for AIMSSEC's work to empower teachers to give children a better education and a better future.

Stage 5
Height 6.5 metres



32

STAGE 1 SIERPINSKI TETRAHEDRON

Study this construction.

What is the shape enclosed inside between the 4 tetrahedra?

Hint: Look at the faces of this shape, both this way up △ and this ▽ and the horizontal faces.

You can make your own models using rolled up newspaper sticks as in these pictures to find out what shape is inside the big tetrahedron made from 4 smaller tetrahedra.

33

Giant balloon pyramid sets new record

34

SIERPINSKI TETRAHEDRON

35

SIERPINSKI NUMBER AND SHAPE PATTERNS INVESTIGATION

Stage 0

Stage 1

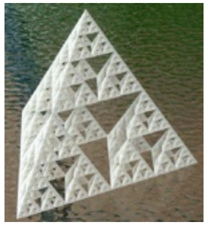
How many small tetrahedra (Stage 0 – 25 cm), were used to make the 6.5 metre high red balloon model tetrahedron?

The green tetrahedron (Stage 1 – 50cm) made from four Stage 0 tetrahedra, has edges of length 50 centimetres in a perfect model. Four Stage 1 tetrahedra are used to make the Stage 2 (1 metre) model, and four of those to make the next one, Stage 3 (2 metres) and so on, and so on.

This white 3D printed model shows the construction of the ideal red balloon model. There are many other questions about scale, number patterns and geometry that you can investigate based on this structure.

For example: 'How many triangular faces are there at each stage?' 'If the (Stage 0) tetrahedra, like the violet tetrahedron above, were solid rather than skeletal, what fraction of the volume of the tetrahedron would be filled at each stage and what fraction would be empty space?'

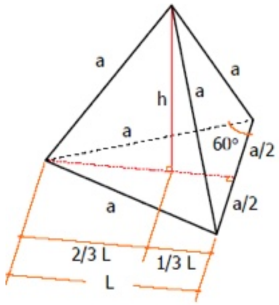
There are many properties to investigate. Different groups in the class might ask different questions and investigate different features.



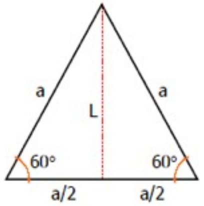
36

SIERPINSKI TETRAHEDRON

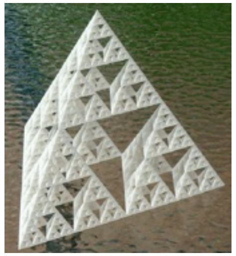
Investigate the construction.



Regular Tetrahedron



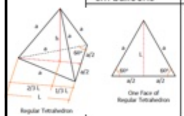
One Face of Regular Tetrahedron



37

SIERPINSKI TETRAHEDRON

Investigate the construction. Fill in the column for Stage 2 then some results



See follow-up slide 57

. Stage	0	1	2	3	4	5	6
Number of tetrahedra	1	4	16	×4 ⟶			4^6=4096
Number of 25 cm balloons	6	24					24576
							Target for new record
Edge length metres a	0.25	0.50		×2 ⟶			16
Altitude of faces. L=a√3/2 metres							13.856
Vertical height h = a√(2/3) metres by Pyth Th.							13.064
Vertical height in feet (×3.28)							42.460

38

SPOT THE PATTERN

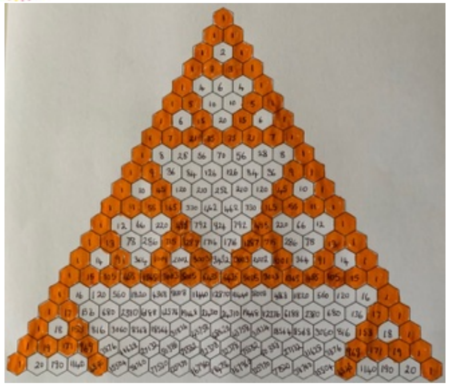
Continue filling in the numbers

Shade the cells containing odd numbers

Later you might try colouring multiples of 2, 3 etc. and you'll discover more patterns.

39

PASCAL'S TRIANGLE



Expand these powers of $(1+x)$
$(1+x)^0$, $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, $(1+x)^4$, $(1+x)^5$.

e.g. $(1+x)^3 = 1 + 3x + 3x^2 + x^3$

$$(1+x)^4 = (1+x)(1+3x+3x^2+x^3)$$
$$= 1 + 4x + 6x^2 + 4x^3 + x^4$$

Can you explain the connection between the process of multiplying out the expansions of $(1+x)^n$ for n = 0, 1, 2, 3, 4 and 5 and the rule for entering numbers in Pascal's triangle?

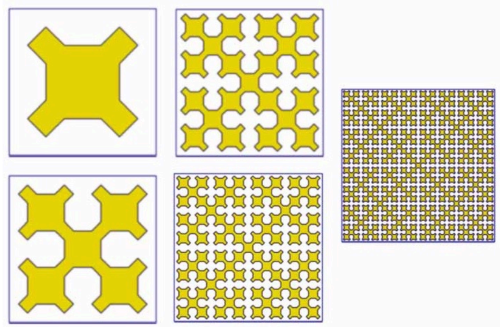
Look at the coefficients of the different powers of x. These are called Binomial Coefficients. Compare these coefficients with the numbers in Pascal's Triangle.

See follow-up slide 58

40

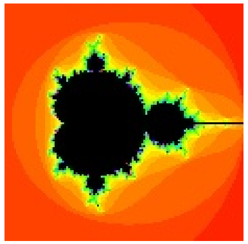
SIERPINSKI'S SPACE FILLING CURVE

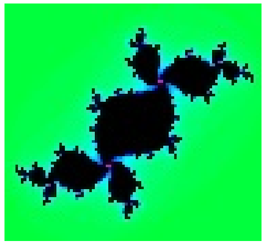
In the limit this fractal curve passes through every interior point of the square.
The area enclosed by the curve is less than half the area of the square.



41

Mandelbrot and Julia Sets





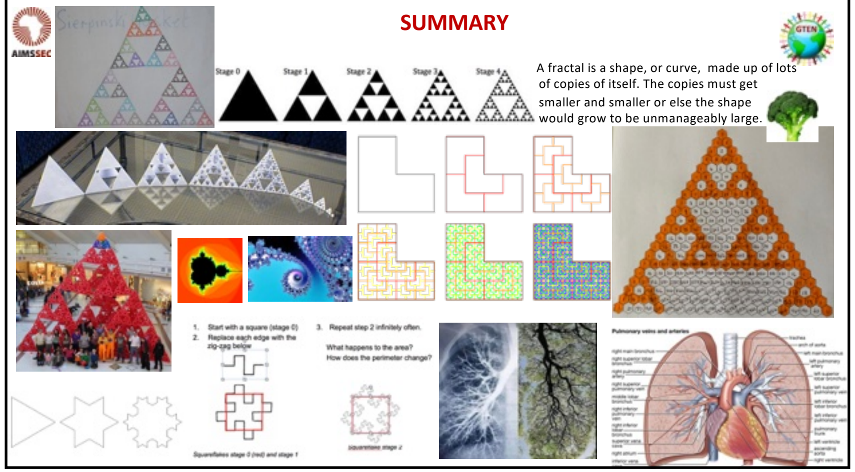
The equation leading to these sets is **F(z) = z² + c**
z is a complex number representing a point in the plane and c is a constant.
This mapping z -> F(z) is repeated over and over again mapping the point z to a sequence of different positions.

For more explanation and to see the sets in detail go to
https://www.youtube.com/watch?v=MAzYWM7Yf4U or just google 'animation of Mandelbrot set'.

42

SUMMARY

A fractal is a shape, or curve, made up of lots of copies of itself. The copies must get smaller and smaller or else the shape would grow to be unmanageably large.



43

LINKS FOR FRACTALS RESOURCES

30-MINUTE FRACTALS LESSON https://aiminghigh.aimssec.ac.za/30-minute-fractals-lesson/

MAKE A VON KOCH POSTER https://aiminghigh.aimssec.ac.za/make-a-von-koch-poster/

THE VON KOCH CURVE https://aiminghigh.aimssec.ac.za/the-von-koch-curve/

SQUAREFLAKE FRACTAL https://aiminghigh.aimssec.ac.za/squareflake-fractal/

SIERPINSKI NUMBER AND SHAPE PATTERNS https://aiminghigh.aimssec.ac.za/sierpinski-number-and-shape-patterns/

THE CHAOS GAME https://aiminghigh.aimssec.ac.za/wp-content/uploads/2019/09/GML19-FRACTALS-The-Chaos-Game.pdf

A free printable net for the **SIERPINSKI TETRAHEDRON** can be found at Think Maths

www.think-maths.co.uk/sites/default/files/2018-01/think_maths_-_sierpinski_tetrahedron_worksheet.pdf

Also have the **MENGER SPONGE** and more at

https://think-maths.co.uk/downloads/building-3d-fractals

Directions for folding the **DRAGON CURVE FRACTAL** https://www.cutoutfoldup.com/216-dragon-curve.php

Wikipedia page https://en.wikipedia.org/wiki/Dragon_curve

JAVA FRACTAL SIMULATIONS/CONSTRUCTIONS https://javalab.org/en/category/math_en/fractal_en/

colouring activities for **PASCAL'S TRIANGLE**

https://www.transum.org/Maths/Activity/Pascals/Triangle.asp?Level=1 OR

http://www.shodor.org/interactivate/activities/ColoringRemainder/

PASCAL'S TRIANGLE in the Number Devil https://prezi.com/ggouikfipb5i/the-number-devil-chapter-7-pascals-triangle/

About the children's book THE NUMBER DEVIL https://en.wikipedia.org/wiki/The_Number_Devil

44

On the AIMING HIGH website there are freely downloadable worksheets, Inclusion and Home Learning Guides with activities for learners of all ages and attainment levels, Notes for Teachers with solutions, Key Questions and Diagnostic Quizzes for formative assessment.

AIMSSEC GTEN YouTube Channel
https://www.youtube.com/c/MathsToys/videos

AIMSSEC FACEBOOK https://www.facebook.com/aimssecsa/

HAPPY MATHS HOUR Weekly on Mondays 17:00 - 18:00 UK time

To apply to join the GTEN Teachers WhatsApp Group and to get information about GTEN or to apply for an AIMSSEC course write to <admin@aimssec.ac.za>

45

THESE LESSON RESOURCES HAVE BEEN DEVELOPED BY UNPAID VOLUNTEERS

THEY ARE FREELY AVAILABLE TO EVERYONE

WE AIM TO GIVE CHILDREN A BETTER EDUCATION AND A CHANCE TO ESCAPE A LIFE OF POVERTY

AIMSSEC supports and empowers teachers in the most seriously deprived communities in Africa. This photo was taken not far from Toni's home in Fish Hoek. Our work helps children living in townships like this.

PLEASE MAKE A DONATION TO SUPPORT THE WORK OF AIMSSEC WHATEVER YOU CAN AFFORD

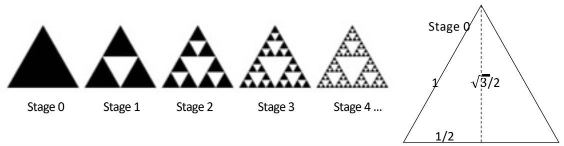
Masiphumelele township near the affluent suburb of Fish Hoek in Cape Town. Photo by Masixole Feni.

Read the article 'What is it like to live in Masiphumelele?
https://www.groundup.org.za/article/what-it-live-masiphumelele_3019/

Please donate to help the work of AIMSSEC
https://www.facebook.com/donate/305379652470708/

46

Workshop Saturday 8 October –
SOLUTIONS AND FOLLOW-UP SIERPINSKI TRIANGLE



SIERPINSKI TRIANGLE					
Stage	Number of shaded triangles	Length of edge of shaded triangles	Area of each shaded triangle	Fraction of area shaded	Fraction unshaded
Stage 0	1	1	$\sqrt{3}/4$	1	0
Stage 1	3	1/2	$(1/2)^2\sqrt{3}/4 = \sqrt{3}/16$	3/4 = 75%	1/4 = 25%
Stage 2	9	1/4	$(1/4)^2\sqrt{3}/4 = \sqrt{3}/64$	9/16 = 56%	7/16 = 44%
Stage 3	27	1/8	$(1/8)^2\sqrt{3}/4 = \sqrt{3}/256$	27/64 = 42%	37/64 = 58%
Stage 4	81	1/16	$(1/16)^2\sqrt{3}/4 = \sqrt{3}/1024$	81/256 = 32%	175/256 = 68%
Stage 5	243	1/32	$(1/32)^2\sqrt{3}/4 = \sqrt{3}/4096$	243/1024 = 24%	781/1024 = 76%

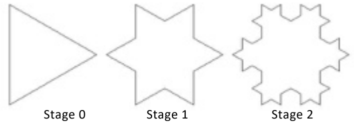
47

SIERPINSKI TETRAHEDRON

STAGE	0	1	2	3	4	5	6
Number of tetrahedra	1	4	4^2=16	4^3=64	4^4=256	4^5=1024	4^6=4096
Number of 25 cm balloons	6	24	96	384	1536	6144	24576
							Target for new record
Edge length a metres	0.25	0.5	1	2	4	8	16
Altitude of triangular face L = a√3/2 metres	0.217	0.433	0.866	1.732	3.464	6.928	13.856
Vertical height H = a√(2/3) metres by Pythagoras Th.	0.204	0.408	0.816	1.633	3.266	6.532	13.064
Vertical height in feet	0.670	1.339	2.679	5.358	10.715	21.430	42.460

48

VON KOCH CURVE

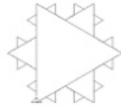


Each line segment is replaced by a zig-zag curve made up of 4 pieces

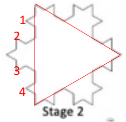
Each piece is one third of the length of the line segment it replaces.

The Von Koch curve is the outer perimeter of the shape formed by repeatedly replacing each edge by the zig-zag curve making smaller and smaller V shapes sticking out from the edges.

If you make a poster by sticking smaller and smaller triangles on the edges, how many extra V shapes will you add to the next stage of the curve?



For all stages, from Stage 2 on, 4 new Vs are added sticking out along each of the 3 edges of stage 0

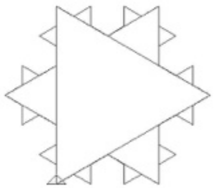


49

MAKING A VON KOCH POSTER – GROUP ACTIVITY

You will need:

- A large backing sheet (flip chart paper is ideal) and glue.
- Triangles drawn accurately and cut from coloured paper
  1 equilateral triangle with sides 27 cm,
  3 equilateral triangles with sides 9 cm,
  12 equilateral triangles with sides 3 cm



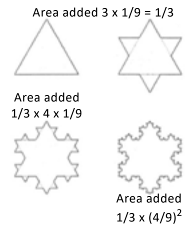
Stick the large triangle on the backing sheet, then stick the 3 triangles with sides 9 cm on the edges, and finally stick the 12 triangles with sides 3 cm on the edges as in the diagram.

The curve at stage 3 can be shown by drawing 48 equilateral triangles with sides of 1 cm or just drawing the 2 edges of the triangles (not shown in this diagram).

Repeating this over and over infinitely often makes the fractal Von Koch curve

50

VON KOCH CURVE
QUESTIONS FOR A MATHS LESSON



How many triangles?

Number of triangles added for stages 1 to 5
= 3 + 3x4 + 3x16 + 3x64 + 3x256
= 3(1 + 4 + 16 + 64 +256)
= 3 x 341 = 1023

How long is the curve?

The length of the curve is increased at each stage by a factor of 4/3 so the length tends to infinity.

How big is the area inside the curve?

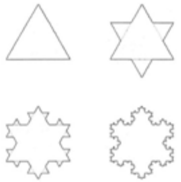
Starting from a triangle area 1 square unit, find the total area added on by summing the infinite geometric series

$$\frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2 + (\frac{4}{9})^3 + (\frac{4}{9})^4 + \cdots]$$

The sum is 3/5 of the area of the original triangle so the area inside the Von Koch curve is 1.6 square units.

51

This mathematics project was used by all the schools in one district of the Eastern Cape for the school-based assessment task in 2011.



The diagram shows the first 4 snowflake iterations of the Von Koch curve.

The edges of the triangle at the first stage are one unit in length.

At each iteration every line segment is replaced by a zig-zag curve made up of 4 pieces, with each piece one third of the length of the line segment replaced.

If this process goes on for ever how long is the curve? Continue filling in this table.

Von Koch snowflake	A Length of edge	B Number of edges	C Perimeter
1	1	3	3
2	⅓	12	4
3			
4			
5			
n			

52

VON KOCH CURVE - LENGTHS

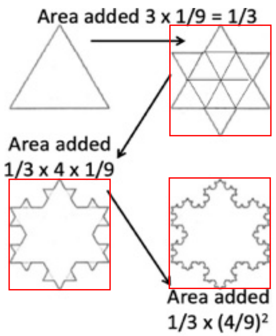
Von Koch snowflake	A Length of edge	B Number of edges	C Perimeter
1	1	3	3
2	⅓	12	4
3	1/9	48	16/3 = 5.3 to 1 d.p.
4	1/27	192	64/9=7.1 to 1 d.p.
5	1/81	768	256/27=12.2 to 1 d.p.
n	$(1/3)^{n-1}$	$3 \times (4)^{n-1}$	$\sum_1^n [\frac{1}{3}^{n-1} \times 3 \times 4^{n-1}]$

53

VON KOCH CURVE - AREA

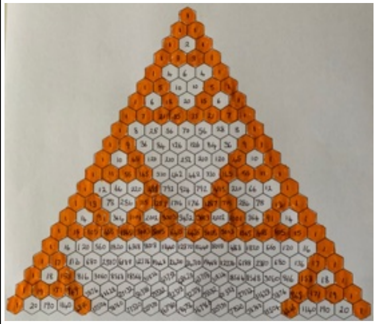
At each stage the triangles drawn on the edge each has an area of $(1/9)^{th}$ of the extra triangle at the previous stage.

The area increases very little but the length of the curve increases at each stage and, in the limit, its length tends to infinity.



54

PASCAL'S TRIANGLE AND THE BINOMIAL EXPANSION



$$(1+x)^0 = 1$$
$$(1+x)^1 = 1 + x$$
$$(1+x)^2 = 1 + 2x + x^2$$
$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$
$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$
$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$
$$(1+x)^6 = \left(1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5\right)(1+x)$$
$$= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$$

$$(1+x)^n = (1+x)(1+x)^{n-1}$$
$$= (1+x)\left(1 + (n-1)x + \frac{(n-1)(n-2)}{2}x^2 + \frac{(n-1)(n-2)(n-3)}{2\times3}x^3 + \ldots\ x^{n-1}\right)$$

55

LET’S PLAY MATHEMATICALLY AND LEARN

Order from AMAZON or TARQUIN https://www.tarquingroup.com/products/aiming-high-lets-play-mathematically

Play Mathematically

- to develop a love for mathematics
- to unlock knowledge and understanding
- to improve numeracy and visualisation skills
- to practise mathematical procedures
- to motivate concentration and critical thinking
- to boost confidence in mathematical ability.

This **first book** in this AIMING HIGH series provides 36 games that are easy to learn and enjoyable to play for any age. Each comes with reflective questions and materials designed to bring out mathematical thinking and provide a deeper understanding of the topic that underlies the game. Even for the youngest players, this can be transformational.

The **second book** offers suggestions for teachers for using games and puzzles in lessons to teach the regular curriculum with different ideas for different age groups.. It is due to be published in mid 2026.

56

AIMS | African Institute for Mathematical Sciences SCHOOLS ENRICHMENT CENTRE

GTEN

Thanks for coming to this workshop.
Use the AIMSSEC ideas on AIMING HIGH and add comments.
Share what you have learned with other teachers.
Try to help all your learners to have a ‘Yes I Can’ attitude to mathematics.

57