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Global Teacher Empowerment Network GTEN
ODDS & EVENS - RED & BLUE GAMES
Saturday 11 September 2021 16.00 – 18.00 London


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AIMS African Institute for Mathematical Sciences SCHOOLS ENRICHMENT CENTRE

MATHS TOYS

Global Teacher Empowerment Network (GTEN)
Programme for Odds Evens Red Blue Games Workshop 11 September 2021

Put your name, country and the age group you teach on the chat

Improve knowledge and understanding of:
Probability
Counting outcomes
Sample space tables

(9) Conjecture and Proof. Proof Sorting Activity.
(8) Triangle Numbers
(7) Isomorphism
(6) Sample space diagrams (2-way tables).
(5) Properties of odd and even numbers.
(4) Odds & Evens Game.
(3) Deciding if a game is fair.
(2) Red Blue Game (5 reds, 5 blues)
(1) Red Blue Game (3 reds, 6 blues)

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LESSON STARTER - RED AND BLUE GAME – any number of players

RESOURCES: You need small objects of two different colours, such as counters or buttons or cards that are the same in all respects other than colour, and a bag (or hat or box or envelope) from which players can draw cards at random.

Put 3 red cards and 6 blue cards in a bag.

In turn, players pick 2 cards at random from the bag, first one which is not replaced, and then another. Players choose to be a 'same' player who wins when both cards are the same colour or a 'different' player who wins when the cards are different colours.

Score points according to whether the cards are the same or different colours.

Then put the cards back in the bag and shake it before the next player takes a turn.

Repeat until one of the players has won 5 points.

Is this a fair game? Is there an equal chance of drawing cards of the same colour as cards of different colours?

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RED BLUE GAME FOR EARLY YEARS AND YEARS 1 AND 2 FREE PLAY

Play the game with 3 red cards and 6 blue cards as described for the starter.

Keep scores.

Play for short sessions as long as the children are enjoying the game.

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RED BLUE GAME FOR YEARS 3 to 5

PLAY FOR FUN THINK TO WIN PLAY TO LEARN

Let your class play and enjoy the Red Blue game, as a lesson starter, in 2 teams, with 3 red and 6 blue cards as described for younger learners.

To discover if it is a fair game, note which team wins each time the class plays the game. Keep a record of the total number of times the cards chosen randomly are the same colour, and the number of times they are different colours. If it is a fair game then, the more often your class plays the game the the closer these two totals will get.

In Year 6 introduce the game with 5 red and 5 blue cards. Learners should investigate whether changing the number of cards of each colour makes a difference. **What do you think?**

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YEARS 6 and 7

IS THIS A FAIR GAME?

Put 5 red cards and 5 blue cards in the bag.

Is there an equal chance of drawing cards of the same colour as cards of different colours?

Answer yes or no.

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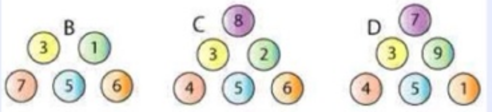
In Game A you pick out two balls at random. If the total is even you win. If it is odd you lose. Is it a fair game?

https://aiminghigh.aimssec.ac.za

ODDS AND EVENS GAMES

Bags B, C and D contain numbered balls for 3 games. You pick out two balls at random from a bag. If the total is even you win. If it is odd you lose. Which set of balls would you choose to have the best chance of winning?



4 games, all played with the same rules.

Are any of the games fair?

Find a set of numbers that gives a fair game.

How do you know which sets of numbers give fair games?

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ADDING ODDS AND EVENS FOR YEARS 6 AND 7

ODD + ODD
Think of any two odd numbers. Add them. Is the sum odd or even?
Think of another two odd numbers. Add them. Is the sum odd or even?
Do this a few more times. **What do you notice?**

EVEN + EVEN
Think of any two even numbers. Add them. Is the sum odd or even?
Think of another two even numbers. Add them. Is the sum odd or even?
Do this a few more times. **What do you notice?**

ODD + EVEN
Think of any two numbers, one odd one even. Add them. Is the sum odd or even?
Think of another two numbers, one odd one even. Add them. Is the sum odd or even?
Do this a few more times. **What do you notice?**

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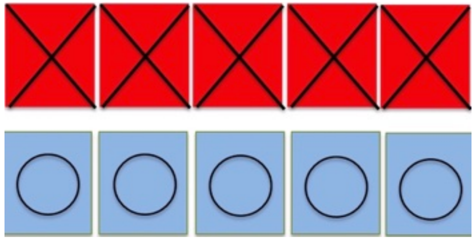
RED BLUE GAME FOR YEARS 6 AND 7

Learners should play and investigate the game with 5 red and 5 blue cards.

The teacher should not give away the **'secret winning strategy'.**



As they play, more and more members of the class will realize for themselves that **they should choose to be 'different'** because the two cards picked are different colours more often than they are the same. They should try to find out why this happens.

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RED BLUE GAME FOR YEAR 7
INVESTIGATING THE RED BLUE GAME WITH 5 CARDS OF EACH COLOUR

The teacher should encourage the class to find out why there is an advantage in choosing to be a different player rather than a same player.

To understand why players are more likely to draw two cards of different colours than two of the same colour, learners need to understand how to fill in a 2-way table showing all possible outcomes.

	R_1	R_2	R_3	R_4	R_5	B_1	B_2	B_3	B_4	B_5
R_1	x									
R_2		x								
R_3			x							
R_4				x						
R_5					x					
B_1						x				
B_2							x			
B_3								x		
B_4									x	
B_5										x

What do the red crosses on the diagonal line tell you?

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RED BLUE GAME INVESTIGATION

What do you notice about the cells (small squares) covered by the yellow triangles?
What do the covered cells represent?
How many are there?

	R_1	R_2	R_3	R_4	R_5	B_1	B_2	B_3	B_4	B_5
R_1	x									
R_2		x								
R_3			x							
R_4				x						
R_5					x					
B_1						x				
B_2							x			
B_3								x		
B_4									x	
B_5										x

What do you notice about the cells (small squares) covered by the grey squares?
What do the covered cells represent?
How many are there?

	R_1	R_2	R_3	R_4	R_5	B_1	B_2	B_3	B_4	B_5
R_1	x									
R_2		x								
R_3			x							
R_4				x						
R_5					x					
B_1						x				
B_2							x			
B_3								x		
B_4									x	
B_5										x

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RED BLUE GAME SAMPLE SPACE DIAGRAM SUMMARY

Explain this 2 way-table and the labels S, D and X in the cells.

There are 40 ways of picking 2 cards the same colour shown by the squares marked S.

There are 50 ways of picking 2 cards of different colours shown by the squares marked D.

CARDS PICKED FIRST

	R_1	R_2	R_3	R_4	R_5	B_1	B_2	B_3	B_4	B_5
R_1	X	S	S	S	S	D	D	D	D	D
R_2	S	X	S	S	S	D	D	D	D	D
R_3	S	S	X	S	S	D	D	D	D	D
R_4	S	S	S	X	S	D	D	D	D	D
R_5	S	S	S	S	X	D	D	D	D	D
B_1	D	D	D	D	D	X	S	S	S	S
B_2	D	D	D	D	D	S	X	S	S	S
B_3	D	D	D	D	D	S	S	X	S	S
B_4	D	D	D	D	D	S	S	S	X	S
B_5	D	D	D	D	D	S	S	S	S	X

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RED BLUE GAME SAMPLE SPACE DIAGRAM

The SAMPLE SPACE is the set of all possible outcomes. In this case it is the set the 90 possible results when you draw 2 cards from a bag with 5 red and 5 blue cards.

	R_1	R_2	R_3	R_4	R_5	B_1	B_2	B_3	B_4	B_5
R_1	X	S	S	S	S	D	D	D	D	D
R_2	S	X	S	S	S	D	D	D	D	D
R_3	S	S	X	S	S	D	D	D	D	D
R_4	S	S	S	X	S	D	D	D	D	D
R_5	S	S	S	S	X	D	D	D	D	D
B_1	D	D	D	D	D	X	S	S	S	S
B_2	D	D	D	D	D	S	X	S	S	S
B_3	D	D	D	D	D	S	S	X	S	S
B_4	D	D	D	D	D	S	S	S	X	S
B_5	D	D	D	D	D	S	S	S	S	X

Each cell shows the result for picking 2 cards at random, first one, then another, from a bag containing 5 red cards and 5 blue cards.

The crosses show you can't pick the same card twice.

There are 90 possibilities: 40 with the same-coloured cards, 50 with different coloured cards. Picking different coloured cards is more likely than the same colour.

The probability of 'same' is $\frac{40}{90} = \frac{4}{9}$.

The probability of 'different' is $\frac{50}{90} = \frac{5}{9}$

It is NOT A FAIR GAME.

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WHAT IS THE SAME AND WHAT IS DIFFERENT ABOUT THESE TWO DIAGRAMS?

RED BLUE GAME SAMPLE SPACE 2-WAY TABLE

	R_1	R_2	R_3	B_1	B_2	B_3	B_4	B_5	B_6
R_1	X	S	S	D	D	D	D	D	D
R_2	S	X	S	D	D	D	D	D	D
R_3	S	S	X	D	D	D	D	D	D
B_1	D	D	D	X	S	S	S	S	S
B_2	D	D	D	S	X	S	S	S	S
B_3	D	D	D	S	S	X	S	S	S
B_4	D	D	D	S	S	S	X	S	S
B_5	D	D	D	S	S	S	S	X	S
B_6	D	D	D	S	S	S	S	S	X

ODD EVEN GAME SAMPLE SPACE 2-WAY TABLE

	O_1	O_2	O_3	E_1	E_2	E_3	E_4	E_5	E_6
O_1	X	E	E	O	O	O	O	O	O
O_2	E	X	E	O	O	O	O	O	O
O_3	E	E	X	O	O	O	O	O	O
E_1	O	O	O	X	E	E	E	E	E
E_2	O	O	O	E	X	E	E	E	E
E_3	O	O	O	E	E	X	E	E	E
E_4	O	O	O	E	E	E	X	E	E
E_5	O	O	O	E	E	E	E	X	E
E_6	O	O	O	E	E	E	E	E	X

For all odd and all even numbers: even + even = even
odd + odd = even
and odd + even = odd

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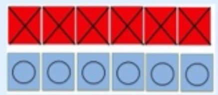
ISOMORPHIC GAMES

https://aiminghigh.aimssec.ac.za/odds-and-evens/

ODDS AND EVENS GAMES

Bags B, C and D contain numbered balls for 3 games. You pick out two balls at random from a bag. If the total is even you win. If it is odd you lose. Which set of balls would you choose to have the best chance of winning?

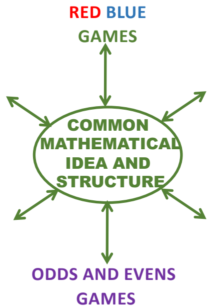
In turn, pick 2 cards at random from the bag, scoring a point (or not) according to whether the cards are the same or different colours.



GAMES IN DISGUISE OR ISOMORPHIC

Can you explain how the games are fair or unfair depending on the number of red or blue cards, or the number of odd or even cards.

Can you explain how outcomes in each of these games correspond to outcomes in the other, how the games have exactly the same mathematical structure?



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RED BLUE GAME SAMPLE SPACE 2-WAY TABLE

	R_1	R_2	R_3	B_1	B_2	B_3	B_4	B_5	B_6
R_1	X	S	S	D	D	D	D	D	D
R_2	S	X	S	D	D	D	D	D	D
R_3	S	S	X	D	D	D	D	D	D
B_1	D	D	D	X	S	S	S	S	S
B_2	D	D	D	S	X	S	S	S	S
B_3	D	D	D	S	S	X	S	S	S
B_4	D	D	D	S	S	S	X	S	S
B_5	D	D	D	S	S	S	S	X	S
B_6	D	D	D	S	S	S	S	S	X

	X								
	●	X							
	●	●	X						
	●	●	●	X					
	●	●	●	●	X				
	●	●	●	●	●	X			
	●	●	●	●	●	●	X		
	●	●	●	●	●	●	●	X	
	●	●	●	●	●	●	●	●	X

This shows a connection between triangle numbers and the number of possible outcomes.

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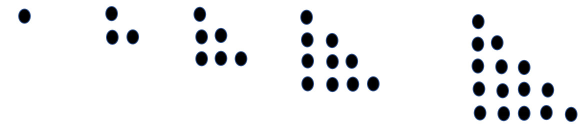
TRIANGLE NUMBERS

1, 3, 6, 10, 15, 21, 28, 36, ...



T_1: 1

T_2: 1 + 2

T_3: 1 + 2 + 3

T_4: 1 + 2 + 3 + 4

T_5: 1 + 2 + 3 + 4 + 5

https://aiminghigh.aimssec.ac.za/triangle-number-picture/

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Clever Carl

Numbers	1	2	3	4	5	...	99	100
Reversed	100	99	98	97	96	...	2	1
Totals	101	101	101	101	101		101	101

$$T_{100} = \frac{1}{2}(100 \times 101) = 5050$$

Give your students the table with many gaps to fill in. Then ask them what they notice about it. Tell them the story and discuss the method.

"What Key Questions would you ask your students?"

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WHAT DO YOU NOTICE ABOUT THIS PICTURE?

$T_6 + T_7$

$7 \times 7 = 49$

Do you think that the sum of two consecutive triangle numbers is always a square number?

Answer yes or no.

Why?

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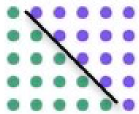
WHAT DO YOU NOTICE IN THIS PICTURE?



Two identical T_5 triangle numbers here make a rectangle.

$2T_5 = 5 \times 6 = 30$ so $T_5 = 15$	Similarly, $2T_{100} = 100 \times 101$ so $T_{100} = 5050$

$T_7 = 1 + 2 + 3 + 4 + 5 + 6 + 7$

$T_7 = 7 + 6 + 5 + 4 + 3 + 2 + 1$

$2T_7 = 8 + 8 + 8 + 8 + 8 + 8 + 8$ so $T_7 = 56/2$

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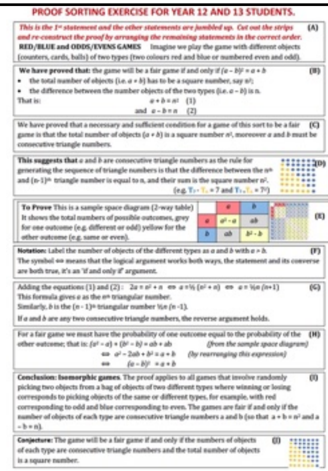
PROOF SORTING EXERCISE FOR YEAR 12 & 13 STUDENTS

Proof requires thinking logically and abstractly often relying on the power of algebra



Proof sorting, or putting a sequence in order, is a good way for learners to make sure that they understand a logical argument.

It can be used for proofs in Euclidean Geometry.

It helps learners to work in pairs or small groups and talk about the ordering.

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Proof Sorting Exercise for Year 12 and 13 Students

This slide, and the next 3, give the statements of the proof in a jumbled order for you to print, cut out, and put in the correct order. See the Inclusion and Home Learning Guide on the AIMING HIGH website for the solution.

This is the 1st statement and the other statements are jumbled up. Cut out the strips and re-construct the proof by arranging the remaining statements in the correct order. **(A)**
RED/BLUE and ODDS/EVENS GAMES Imagine we play the game with different objects (counters, cards, balls) of two types (two colours red and blue or numbered even and odd).

We have proved that: the game will be a fair game if and only if $(a - b)^2 = a + b$ **(B)**
- the total number of objects (i.e. $a + b$) has to be a square number, say n^2;
- the difference between the number objects of the two types (i.e. $a - b$) is n.

That is: $a + b = n^2$ (1)
and $a - b = n$ (2)

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Proof Sorting Exercise for Year 12 and 13 Students

We have proved that a necessary and sufficient condition for a game of this sort to be a fair game is that the total number of objects $(a + b)$ is a square number n^2, moreover a and b must be consecutive triangle numbers. **(C)**

This suggests that a and b are consecutive triangle numbers as the rule for generating the sequence of triangle numbers is that the difference between the n^{th} and $(n-1)^{th}$ triangle number is equal to n, and their sum is the square number n^2. (e.g. $T_7 - T_6 = 7$ and $T_7 + T_6 = 7^2$) **(D)**

To Prove This is a sample space diagram (2-way table) It shows the total numbers of possible outcomes, grey for one outcome (e.g. different or odd) yellow for the other outcome (e.g. same or even). **(E)**

	a	b
a	$a^2 - a$	ab
b	ab	$b^2 - b$

Notation: Label the number of objects of the different types as a and b with $a > b$. **(F)**
The symbol ⇔ means that the logical argument works both ways, the statement and its converse are both true, it's an 'if and only if' argument.

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Proof Sorting Exercise for Year 12 and 13 Students

Adding the equations (1) and (2) : $2a = n^2 + n \Leftrightarrow a = ½(n^2 + n) \Leftrightarrow a = ½n(n+1)$ **(G)**
This formula gives a as the n^{th} triangular number.
Similarly, b is the $(n - 1)^{th}$ triangular number $½n(n - 1)$.
If a and b are any two consecutive triangle numbers, the reverse argument holds.

For a fair game we must have the probability of one outcome equal to the probability of the other outcome; that is: $(a^2 - a) + (b^2 - b) = ab + ab$ *(from the sample space diagram)* **(H)**
$\Leftrightarrow a^2 - 2ab + b^2 = a + b$ *(by rearranging this expression)*
$\Leftrightarrow (a - b)^2 = a + b$

Conclusion: Isomorphic games. The proof applies to all games that involve randomly picking two objects from a bag of objects of two different types where winning or losing corresponds to picking objects of the same or different types, for example, with red corresponding to odd and blue corresponding to even. The games are fair if and only if the number of objects of each type are consecutive triangle numbers a and b (so that $a + b = n^2$ and $a - b = n$). **(I)**

Conjecture: The game will be a fair game if and only if the numbers of objects of each type are consecutive triangle numbers and the total number of objects is a square number. **(J)**

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For the ODDS AND EVENS GAMES Learning Pack see:

https://aiminghigh.aimssec.ac.za/odds-and-evens/

LEARNING PACKS CONTAIN:

1. Worksheets
2. Templates and instructions for making resources
3. Notes for Teachers with
   - solutions
   - curriculum links
   - suggestions for teaching
   - Key Questions to guide learning
4. Inclusion and Home Learning Guides with
   - a starter activity for a mixed–age group to do together
   - a collection of learning activities to suit all ages from 4 to 18+.

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Global Teacher Empowerment Network (GTEN)

NEW SKILLS NEW HOPES NEW HORIZONS
for teachers and learners worldwide
PROBABILITY RESOURCES

The activity **In a Box** https://aiminghigh.aimssec.ac.za/in-a-box/ offers another context for exploring exactly the same game and underlying mathematical structure. Use it as a follow-up a few weeks after working on Odds & Evens.

Also see **Special Sums** https://aiminghigh.aimssec.ac.za/special-sums/
Red or Black: https://aiminghigh.aimssec.ac.za/red-or-black-game/
Nines or Tens https://aiminghigh.aimssec.ac.za/nines-and-tens/
Two Aces https://aiminghigh.aimssec.ac.za/two-aces/
In the Bag https://aiminghigh.aimssec.ac.za/in-the-bag/
Twos Company https://aiminghigh.aimssec.ac.za/twos-company/
Same Sweets https://aiminghigh.aimssec.ac.za/same-sweets/

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LET'S PLAY MATHEMATICALLY AND LEARN

Order from AMAZON or TARQUIN https://www.tarquingroup.com/products/aiming-high-lets-play-mathematically

Play Mathematically
- to develop a love for mathematics
- to unlock knowledge and understanding
- to improve numeracy and visualisation skills
- to practise mathematical procedures
- to motivate concentration and critical thinking
- to boost confidence in mathematical ability.

This **first book** in this AIMING HIGH series provides 36 games that are easy to learn and enjoyable to play for any age. Each comes with reflective questions and materials designed to bring out mathematical thinking and provide a deeper understanding of the topic that underlies the game. Even for the youngest players, this can be transformational.

The **second book** offers suggestions for teachers for using games and puzzles in lessons to teach the regular curriculum with different ideas for different age groups.. It is due to be published in mid 2026.

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