

Global Teacher Empowerment Network GTEN
 Saturday 28 October 2023 16.00 – 18.00 London Time
QUADRATIC FUNCTIONS

circle
ellipse
parabola
hyperbola

Learn why the flight path of a projectile is a parabola.

Toni Beardon **Caroline Ainslie** **Sam Okoth**

$y = x^2$ $y = 2x^2$ $y = \frac{1}{2}x^2$ $y = ax^2 + bx + c$

GRAPHING WORKOUT

Focussing a Beam of Light

1

AIMS African Institute for Mathematical Sciences
 SCHOOLS ENRICHMENT CENTRE

MATHS TOYS

Global Teacher Empowerment Network (GTEN)

PROGRAMME: QUADRATIC FUNCTIONS, EQUATIONS AND GRAPHS

Learning Spiral

IMPROVE SKILLS, KNOWLEDGE AND UNDERSTANDING OF:

UPPER SECONDARY
 LOWER SECONDARY
 UPPER PRIMARY
 LOWER PRIMARY
 EARLY YEARS

9. Review of the workshop
 8. Why the flight path of a projectile is a parabola
 7. Transformations of graphs
 6. Matching graphs to their equations and critical points on the graph.
 5. Quadratic functions, their graphs and quadratic equations
 4. Applications to telescopes and searchlights.
 3. Conic sections
 2. Workout using body and arms to act out graphs of functions.
 1. Noticing the flight paths of objects in sports and in firing missiles.

2

GTEN

Learner **Teacher**

AIMSSEC

During this session you should wear 2 hats. Do the activities, as if you were a learner, to appreciate how a learner would do them.

Reflect as a teacher on what they could learn and how you could guide the learning

When you see this icon, answer the question pausing the video if you need the time.

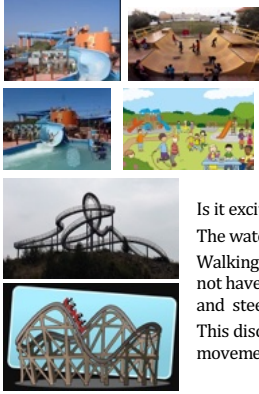
3

UPS AND DOWNS * NOTICE THE FLIGHT PATHS

Umpire, Bowler, Ball, Batsman, Wicket, Fielders

4

UPS AND DOWNS – CURVES AND SLOPES



Talk about the UPS and DOWNS of rides on water slides, roller coasters, playground slides and skateboard ramps. The children should look at the pictures and describe the rides.

What are the similarities in the shapes of water slides and roller coasters?

What are the differences? Does a ride usually start at the top or at the bottom, and is this a difference between water slide and roller coaster rides? Why is that?


Is it exciting when a ride climbs steeply or drops steeply (suddenly)?

The water slides are actually in Muizenberg where the AIMSSEC office is located. Walking up or down hills is part of everyday experience so, although they might not have experienced rides like this, the discussion introduces ideas of curves and steepness (gradient) which are important concepts in mathematics. This discussion will help children to develop a vocabulary about curves and movement on curved tracks, that will be used later in describing graphs.

5

UPS AND DOWNS – CURVES AND SLOPES

With your children use scrap cardboard to make a marble run or a ramp for toy cars. The one illustrated on the left is made from paper plates and the cylinder from inside a roll of kitchen paper. You need scissors, sticky tape and imagination and you can improvise with other scrap materials.



To help your young learners with their mathematics in school when they are older, talk about the spiral shapes, curves and slopes and how steep the slopes are. Can you even make a run that has some ups as well as downs.

You could do experiments. Time how long it takes for a marble or car to travel down the ramp from different heights, and how far it goes on different surfaces once it reaches the bottom.

6

GRAPHS WORKOUT - LINES

Talk about which graphs the stick men demonstrate, how and why the arm positions show different lines, and what the equations could be.

All face the front and follow the leader who has his or her back to the group. The leader defines the x- and y- axes, then calls out equations in turn. Everyone uses their arms to make the graph for that equation.

Start with easy equations, for example:

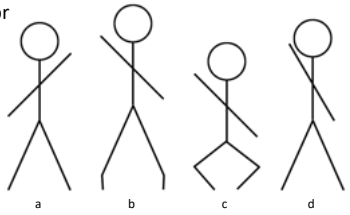

$y = x$, $y = -x$, $y = 2x$, $y = -2x$,
 $y = -x + 3$ (stand on tiptoe!),
 $y = -x - 2$ (crouch down),
 $y = -2x - 2$ (still crouching down)...

Talk about the graphs and how you have acted them out.

For a challenge try: $y = 2$ (easy), $x = 2$, $x + y = 3$...

Let other members of the group act as the leader.

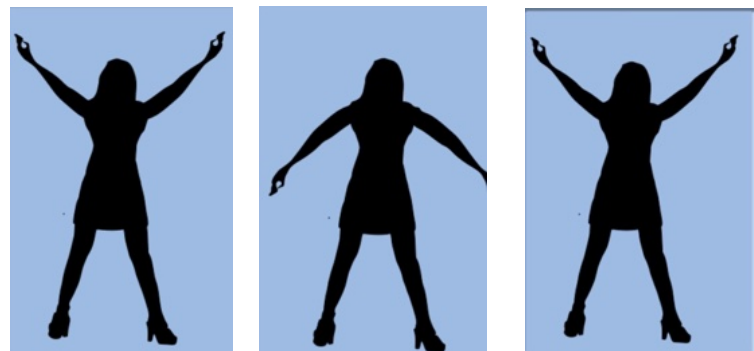
What lines are these stick people demonstrating?


7


GRAPHS WORKOUT * THE MODULUS FUNCTION


Mod x : $y = |x|$ $y = -|x|$ $y = |-x|$




8


AIMSSEC **GRAPHS WORKOUT * QUADRATIC FUNCTIONS (PARABOLAS)** 

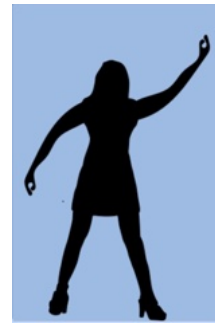
$y = x^2$



$y = 2x^2$



$y = \frac{1}{2}x^2$


9


AIMSSEC **GRAPH WORKOUT * CUBIC FUNCTIONS AND CIRCLES** 

$y = x^3$


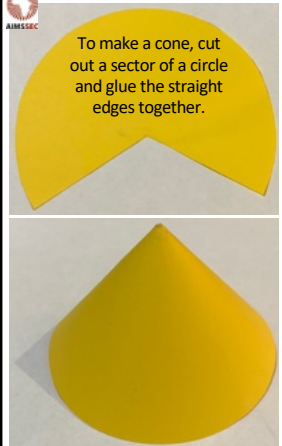
$y = -x^3$


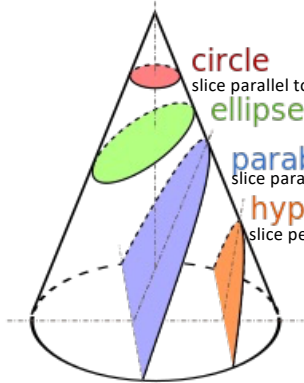
$x^2 + y^2 = 4$


10

AIMSSEC **CONIC SECTIONS** 

To make a cone, cut out a sector of a circle and glue the straight edges together.

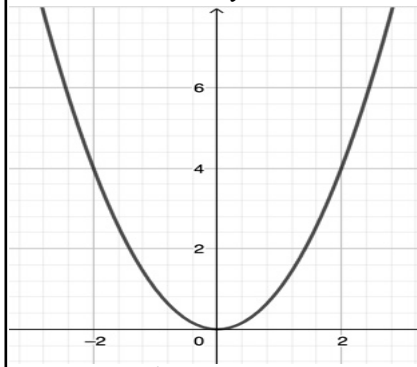


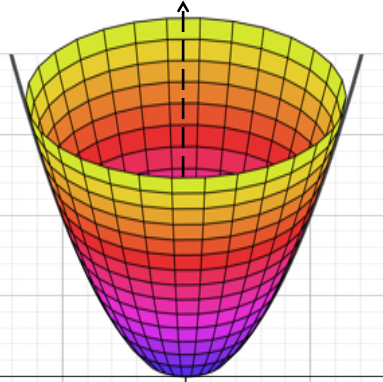


- circle**
slice parallel to base
- ellipse**
slice parallel to sloping edge
- parabola**
slice parallel to sloping edge
- hyperbola**
slice perpendicular to base

11

Parabola $y = x^2$





Rotating a parabola about its axis gives a paraboloid

Illustration of paraboloid by Krishnavedala - Own work, CC BY-SA 3.0, <https://commons.wikimedia.org/w/index.php?curid=15278826>

12

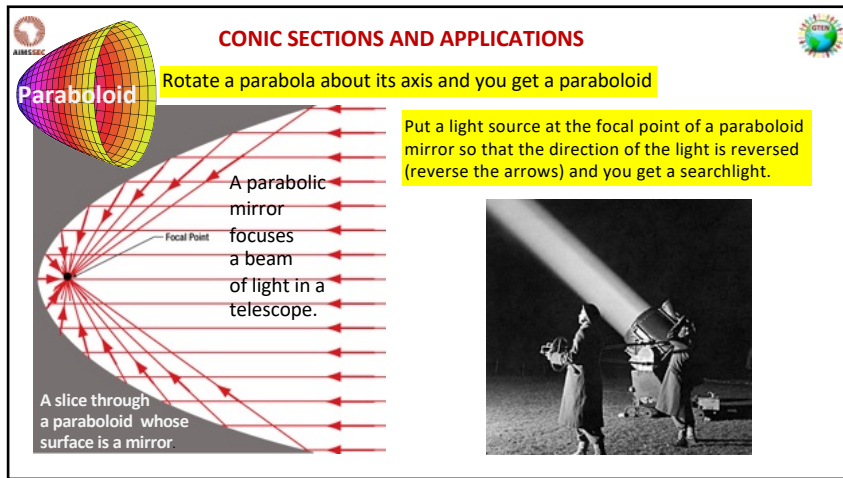
CONIC SECTIONS AND APPLICATIONS

Paraboloid Rotate a parabola about its axis and you get a paraboloid

Put a light source at the focal point of a paraboloid mirror so that the direction of the light is reversed (reverse the arrows) and you get a searchlight.

A parabolic mirror focuses a beam of light in a telescope.

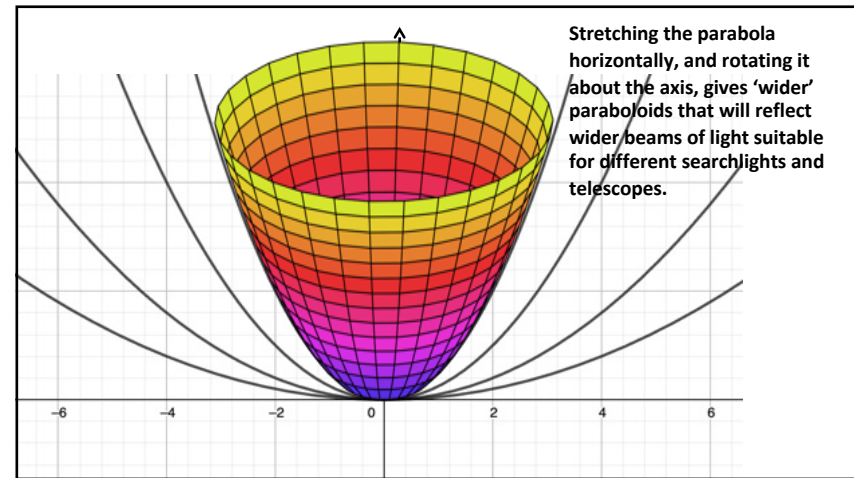
A slice through a paraboloid whose surface is a mirror.



The diagram shows a 3D paraboloid on the left and a searchlight on the right. Red arrows represent light rays originating from a focal point and reflecting off the paraboloid's surface. The searchlight image shows a person holding a large parabolic mirror that reflects a bright beam of light.

13

Stretching the parabola horizontally, and rotating it about the axis, gives 'wider' paraboloids that will reflect wider beams of light suitable for different searchlights and telescopes.



The graph shows a wide, shallow paraboloid opening upwards on a coordinate plane. The x-axis ranges from -6 to 6, and the y-axis is vertical. The paraboloid is colored with a gradient from purple at the bottom to yellow at the top.

14

QUADRATIC MATCHING

<https://aiminghigh.aimssec.ac.za/quadratic-matching-1/>

EXAMPLE

$y = 11x - 5 - 2x^2$

$y = (2x - 1)(5 - x)$

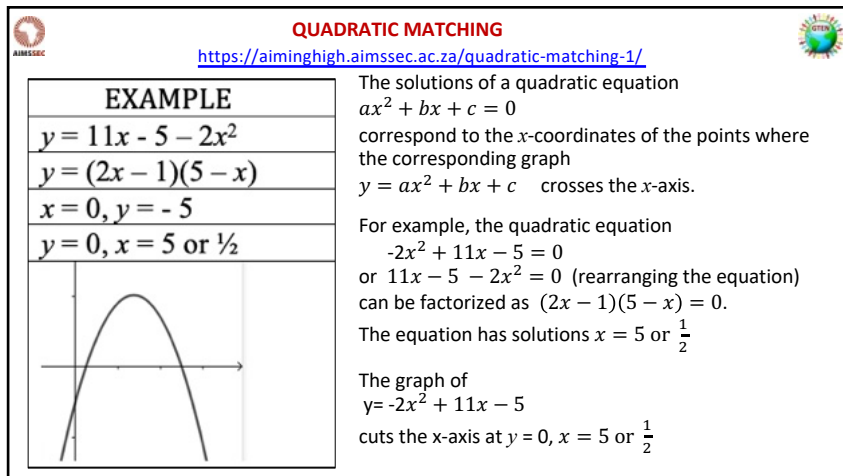
$x = 0, y = -5$

$y = 0, x = 5 \text{ or } \frac{1}{2}$

The solutions of a quadratic equation $ax^2 + bx + c = 0$ correspond to the x -coordinates of the points where the corresponding graph $y = ax^2 + bx + c$ crosses the x -axis.

For example, the quadratic equation $-2x^2 + 11x - 5 = 0$ or $11x - 5 - 2x^2 = 0$ (rearranging the equation) can be factorized as $(2x - 1)(5 - x) = 0$. The equation has solutions $x = 5$ or $\frac{1}{2}$.

The graph of $y = -2x^2 + 11x - 5$ cuts the x -axis at $y = 0, x = 5$ or $\frac{1}{2}$



The graph shows a downward-opening parabola on a coordinate plane. The x-axis has tick marks at 0, 1, 2, 3, 4, 5, and 6. The parabola crosses the x-axis at x = 0.5 and x = 5, and the y-axis at y = -5.

15

QUADRATIC MATCHING - INTERCEPTS WITH AXES

Match each of the graphs to the points where it cuts the x -axis, that is the line $y = 0$. For example, G1 matches with X5 and Y2.

G1	G2	G3	G4	G5	G6	G7
Y1	Y2	Y3	Y4	Y5	Y6	Y7
X1	X2	X3	X4	X5	X6	X7

Sketch the graph of $y = x^2$ here

$x = 0, y = -16$	$x = 0, y = 16$	G1	X5	Y2
$x = 0, y = 16$	$x = 0, y = -8$	G2		Y2
$x = 0, y = 8$	$x = 0, y = 8$	G3		Y3
$x = 0, y = -16$	$y = 0, x = 8 \text{ or } -2$	G4		Y4
$y = 0, x = -4 \text{ or } -2$	$y = 0, x = -2 \text{ or } 4$	G5		Y4
$y = 0, x = 4 \text{ or } 2$	$y = 0, x = 8 \text{ or } 2$	G6		Y1
$y = 0, x = 4$	$y = 0, x = -8 \text{ or } 2$	G7		Y1

16

QUADRATIC MATCHING

Match each of the graphs to its equation both in its quadratic polynomial form (A cards) and in its factorized form (B cards). For example, the graph G1 cuts the x-axis at two positive values of x, two positive roots or solutions. G1 matches with A5 and B5.

G1	G2	A1 $y = x^2 + 6x - 16$	A2 $y = x^2 - 8x + 16$	G1 A5 B5
G3	G4	A3 $y = 8 - x^2 + 2x$	A4 $y = 6x - x^2 - 8$	G2
G5	G6	A5 $y = x^2 - 10x + 16$	A6 $y = x^2 + 6x + 8$	G3
G7		A7 $y = x^2 - 6x - 16$	B1 $y = (x - 8)(x + 2)$	G4
		B2 $y = (x + 4)(x + 2)$	B3 $y = (x + 2)(4 - x)$	G5
		B4 $y = (x - 4)(2 - x)$	B5 $y = (x - 8)(x - 2)$	G6
		B6 $y = (x - 4)(x - 4)$	B7 $y = (x + 8)(x - 2)$	G7

Sketch the graph of $y = x^2$ here

17

QUADRATIC MATCHING - INTERCEPTS WITH AXES

Match each of the graphs to the points where it cuts the axes. For example G1 matches with X4 and Y4.

G1	G2	Y1 $x = 0, y = -16$	Y2 $x = 0, y = 16$	G1 X5 Y2
G3	G4	Y2 $x = 0, y = 16$	Y3 $x = 0, y = -8$	G2 X6 Y2
G5	G6	Y4 $x = 0, y = 8$	Y4 $x = 0, y = 8$	G3 X4 Y3
G7		Y1 $x = 0, y = -16$	X1 $y = 0, x = 8 \text{ or } -2$	G4 X2 Y4
		X2 $y = 0, x = -4 \text{ or } -2$	X3 $y = 0, x = -2 \text{ or } 4$	G5 X3 Y4
		X4 $y = 0, x = 4 \text{ or } 2$	X5 $y = 0, x = 8 \text{ or } 2$	G6 X1 Y1
		X6 $y = 0, x = 4$	X7 $y = 0, x = -8 \text{ or } 2$	G7 X7 Y1

Sketch the graph of $y = x^2$ here

18

QUADRATIC MATCHING

Match each of the graphs to its equation both in its factorized form and its quadratic polynomial form. For example, the graph G1 cuts the x-axis at two positive values of x, two positive roots or solutions. G1 matches with A5 and B5.

G1	G2	A1 $y = x^2 + 6x - 16$	A2 $y = x^2 - 8x + 16$	G1 A5 B5
G3	G4	A3 $y = 8 - x^2 + 2x$	A4 $y = 6x - x^2 - 8$	G2 A2 B6
G5	G6	A5 $y = x^2 - 10x + 16$	A6 $y = x^2 + 6x + 8$	G3 A4 B4
G7		A7 $y = x^2 - 6x - 16$	B1 $y = (x - 8)(x + 2)$	G4 A6 B2
		B2 $y = (x + 4)(x + 2)$	B3 $y = (x + 2)(4 - x)$	G5 A3 B3
		B4 $y = (x - 4)(2 - x)$	B5 $y = (x - 8)(x - 2)$	G6 A7 B1
		B6 $y = (x - 4)(x - 4)$	B7 $y = (x + 8)(x - 2)$	G7 A1 B7

Sketch the graph of $y = x^2$ here

19

QUADRATIC MATCHING SUMMARY

$y = x^2 + 6x - 16$	G7 A1	$y = x^2 - 8x + 16$	G2 A2	$y = 8 - x^2 + 2x$	G5 A3	$y = 6x - x^2 - 8$	G3 A4
$y = (x + 8)(x - 2)$	B7	$y = (x - 4)(x - 4)$	B6	$y = (x + 2)(4 - x)$	B3	$y = (x - 2)(4 - x)$	B4
$x = 0, y = -16$	Y1	$x = 0, y = 16$	Y2	$x = 0, y = 8$	Y4	$x = 0, y = -8$	Y3
$y = 0, x = -8 \text{ or } 2$	X7	$y = 0, x = 4$	X6	$y = 0, x = -2 \text{ or } 4$	X3	$y = 0, x = 2 \text{ or } 4$	X4

EXAMPLE

$y = x^2 - 10x + 16$	G1 A5	$y = x^2 + 6x + 8$	G4 A6	$y = x^2 - 6x - 16$	G6 A7	$y = 11x - 5 - 2x^2$
$y = (x - 2)(x - 8)$	B5	$y = (x + 2)(x + 4)$	B2	$y = (x + 2)(x - 8)$	B1	$y = (2x - 1)(5 - x)$
$x = 0, y = 16$	Y2	$x = 0, y = 8$	Y4	$x = 0, y = -16$	Y1	$x = 0, y = -5$
$y = 0, x = 2 \text{ or } 8$	X5	$y = 0, x = -2 \text{ or } -4$	X2	$y = 0, x = -2 \text{ or } 8$	X1	$y = 0, x = 5 \text{ or } \frac{1}{2}$

20

TRANSFORMATIONS OF GRAPHS

$y = f(x) + a$ translation $y = x^2$	$y = -f(x)$ reflection in x - axis $y = x^2$	$y = kf(x)$ stretch in y - axis scale factor k $y = x^2$
$y = f(x - a)$ plus a - left minus a - right opposite to what you might think! $y = x^2$	$y = f(-x)$ reflection in y - axis $y = x^2$	$y = f(kx)$ stretch in x - axis scale factor 1/k $y = x^2$

21

TRANSFORMATIONS OF GRAPH OF $y = x^2$ AND SOLUTIONS OF QUADRATIC EQUATIONS

The family of graphs of quadratic functions (parabolas) are all transformations of the graph of $y = x^2$ (the parent graph).

To find out how the parent graph is transformed to give the graph of the quadratic function $y = ax^2 + bx + c$ the equation can be written as $y = a(x + p)^2 + q$ where $p = \frac{b}{2a}$ and $q = c - \frac{b^2}{4a}$.

This diagram shows how the graph of $y = x^2$ is **stretched** in the y direction by a factor a , and **translated** p units in the positive x direction and q units in the y direction.

22

UPS AND DOWNS – Equations of Motion using calculus

The horizontal and vertical distances travelled are x and y , and g the constant of acceleration vertically downwards due to gravity. The initial horizontal velocity is u metres per second and the initial vertical velocity is v metres per second upwards. The x -coordinate and y -coordinate of the object at time t are:

$$x = ut \quad (1) \quad (\text{distance} = \text{velocity} \times \text{time})$$

$$y = vt - \frac{1}{2}gt^2 \quad (2)$$

Differentiate equations (1) and (2) to find the velocity

$$\frac{dx}{dt} = u \quad (\text{Horizontal velocity - rate of change of position})$$

$$\frac{dy}{dt} = v - gt \quad (\text{Vertical velocity - rate of change of position})$$

Differentiate again to find the acceleration

$$\frac{d^2x}{dt^2} = 0 \quad (\text{Horizontal acceleration - rate of change of velocity})$$

$$\frac{d^2y}{dt^2} = -g \quad (\text{Vertical acceleration - rate of change of velocity})$$

23

UPS AND DOWNS – Why the flight path (trajectory) of a projectile is a parabola

We take x and y to be the horizontal and vertical distances travelled and g the constant of acceleration vertically downwards due to gravity.

If the initial horizontal velocity is u metres per second then the x coordinate of the object at time t is $x = ut$ (1).

If the initial vertical velocity is v metres per second upwards, then the y coordinate of the object at time t is $y = vt - \frac{1}{2}gt^2$ (2).

These equations are given by Newton's Laws of Motion. Even if you don't understand the physics and calculus involved, it's easy to see why these equations give a parabolic flight path.

From equation (1): $t = ?$

To eliminate t and get the equation of the flight path, substitute this value of t in equation (2): $y = ?$

24

UPS AND DOWNS – Why the flight path of a projectile is a parabola

Take x and y to be the horizontal and vertical distances travelled and g the constant of acceleration due to gravity vertically downwards. If the initial horizontal velocity is u metres per second, then the x coordinate of the object at time t is $x = ut$ (1).

If the initial vertical velocity is v metres per second upwards, then the y coordinate of the object at time t is $y = vt - \frac{1}{2}gt^2$ (2).

Eliminate t to get the equation of the flight path. From (1): $t = \frac{x}{u}$

By substituting this value of t in equation (2): $y = \frac{v}{u}x - \frac{g}{2u^2}x^2$

This is a quadratic equation. It is the equation of a parabola $y = ax^2 + bx + c$

with the constants $a = -\frac{g}{2u^2}$, $b = \frac{v}{u}$ and $c = 0$.

25

REVIEW

Equations of motion

$$x = ut$$

$$y = vt - \frac{1}{2}gt^2$$

Parent graph $y = x^2$

$$y = ax^2 + bx + c$$

Key features of a parabola: $x = -b/2a$, $y = a(x - b/2a)^2$, $(-b/2a, 0)$, $(0, 0)$

Geometric shapes: circle, ellipse, parabola, hyperbola

Focussing a Beam of Light Telescope

GRAPHING WORKOUT

26

AIMINGHIGH TEACHER NETWORK

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See the Learning Packs on AIMING HIGH

QUADRATIC FUNCTIONS <https://aiminghigh.aimssec.ac.za/quadratic-functions/>

QUADRATIC MATCHING 1 AND 2 <https://aiminghigh.aimssec.ac.za/quadratic-matching-1/>
<https://aiminghigh.aimssec.ac.za/quadratic-matching-2/>

GRAPHING QUADRATIC EQUATIONS
<https://aiminghigh.aimssec.ac.za/graphing-quadratic-equations/>

QUADRATIC EQUATIONS <https://aiminghigh.aimssec.ac.za/quadratic-equations/>

ANIMATION OF PROJECTILE MOTION <https://phet.colorado.edu/en/simulations/projectile-motion>

Most of the 'Let's Play Mathematically' games are on AIMING HIGH so play the games with your learners, and join the discussion to win <https://www.tarquingroup.com/products/aiming-high-lets-play-mathematically>

Interactive demonstration of projectile motion <https://phet.colorado.edu/en/simulations/projectile-motion>

27

LET'S PLAY MATHEMATICALLY AND LEARN

Order from AMAZON or TARQUIN <https://www.tarquingroup.com/products/aiming-high-lets-play-mathematically>

Play Mathematically

- to develop a love for mathematics
- to unlock knowledge and understanding
- to improve numeracy and visualisation skills
- to practise mathematical procedures
- to motivate concentration and critical thinking
- to boost confidence in mathematical ability.

This **first** book in this AIMING HIGH series provides 36 games that are easy to learn and enjoyable to play for any age. Each comes with reflective questions and materials designed to bring out mathematical thinking and provide a deeper understanding of the topic that underlies the game. Even for the youngest players, this can be transformational.

The **second** book offers suggestions for teachers for using games and puzzles in lessons to teach the regular curriculum with different ideas for different age groups.. It is due to be published in mid 2026.

28