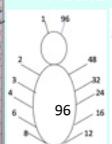


Global Teacher Empowerment Network GTEN
 Saturday 11 FEBRUARY 2023 16.00 – 18.00 London Time

HOW MANY FACTORS?

Multiples of

SIEVE OF ERATOSTHENES

MULTIPLES, CYCLES & MODULAR ARITHMETIC

HCF & LCM

$\frac{1}{30} + \frac{1}{96} = \frac{16+5}{2^5 \times 3 \times 5} = \frac{21}{480} = \frac{7}{160}$

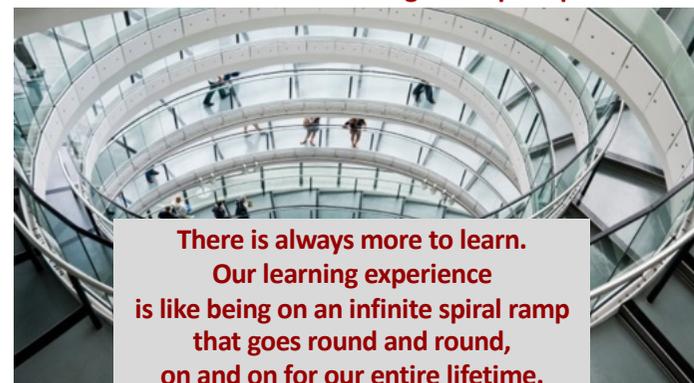
$\sqrt{\frac{1008}{847}}$

Toni Beardon Caroline Ainslie Cynthia Fries

How do you find the total number of factors without listing them?
 For example:
 15552 = 2⁶ × 3⁵
 has 7 × 6 = 42 factors.
 Explain.

1

LIFELONG LEARNING
 Build skills and understanding on a spiral path



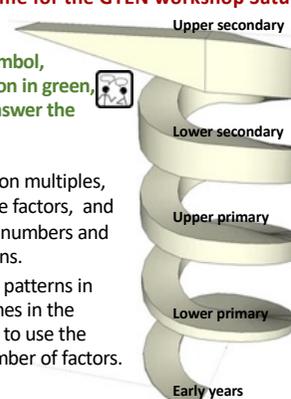
**There is always more to learn.
 Our learning experience
 is like being on an infinite spiral ramp
 that goes round and round,
 on and on for our entire lifetime.**

2

HOW MANY FACTORS?
 Programme for the GTEN workshop Saturday 11 February 2023

When you see this symbol, and you see a question in green, do the activity and answer the question.

Today we are working on multiples, factorization into prime factors, and HCF and LCM both for numbers and for algebraic expressions. We'll investigate some patterns in the powers of the primes in the factorization, and how to use the powers to find the number of factors.



- Upper secondary 10. Investigation of numbers of factors and the powers in the prime factorization.
- 9. Algebraic factors and fractions
- Lower secondary 8. Multiples, cycles and clock (modular) arithmetic
- 7. HCF and LCM applied to addition of fractions
- Upper primary 6. Prime factorisation
- 5. The Factor Ladder
- 4. The Factor Tree
- Lower primary 3. The Factor Bug
- 2. Finding prime numbers by the sieve method
- Early years 1. Patterns in multiples.

3

HOW MANY FACTORS
DO THE NUMBERS 2100 AND 3100 HAVE?

In this workshop we'll see how ideas of multiples, factors, prime numbers, and HCF and LCM, both in number and algebra, are learned in school and how the answer to the question "How Many Factors?" depends on the Fundamental Counting Principle that underpins probability.

The prime factorization of 2100 is 2² × 3 × 5² × 7 and it has 3 × 2 × 3 × 2 = 36 factors.

The prime factorization of 3100 is 2² × 5² × 31 and it has 3 × 3 × 2 = 18 factors.



4

MULTIPLE PATTERNS

<https://aiminghigh.aimssec.ac.za/multiple-patterns>




Shade multiples on a 100-square, describe the patterns and explain why they occur.

Example:
Multiples of 6

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

<https://aiminghigh.aimssec.ac.za/multiple-patterns/>

5

MULTIPLE PATTERNS






Choose one grid.
Shade all the squares with multiples of the given number.

If this is done in groups in a lesson then each person could choose a different set of multiples so the group can compare their different patterns.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 2									
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 3									
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 4									
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

6

MULTIPLE PATTERNS




1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 2									
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 3									
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 4									
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

7

PRIME SIEVE – sieve of Eratosthenes.

<https://aiminghigh.aimssec.ac.za/prime-sieve/>

Amazing Maths YouTube video https://youtu.be/2l2EcvDLA_o






On the 100 square grid, circle the number 2 and make a line through 4, 6, 8, 10 and the rest of the 2 times table.

Circle the number 3 and make a line through 6, 9, 12, 15 and all the rest of the 3 times table.

Do the same for the 5 and 7 times tables.

What do you notice about the multiples of 8, 9 and 10?

What can you say about the numbers that are not crossed out?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

8



PRIME SIEVE

<https://aiminghigh.aimssec.ac.za/years-6-9-prime-sieve/>

2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

In a 100 square shade 1, then not 2, 3, 5 or 7 but all their multiples. You are left with all the prime numbers up to 100. Do it. Then explain why it works.



Why is it that you only need to cross out multiples of 7 to complete the sieve process up to 100, and multiples up to 13 to get all the primes up to 200?

If a number $n = p \times q$ then p or q must be $\leq \sqrt{n}$.

9



DIAGNOSTIC QUIZ



Diagnostic Questions

These numbers are written as a product of prime factors.

Which of them is a square number?

A

B

C

D

$2^4 \times 3$

$3^2 \times 5^4$

$2^3 \times 5^2$

$2 \times 3^2 \times 5^4$

The correct answer is : **B**

$3^2 \times 5^4 = (3 \times 5^2)^2$

A. Wrong – for a square number, the powers of all the factors must be even

C. & D. 2 occurs to an odd power so these are not square numbers

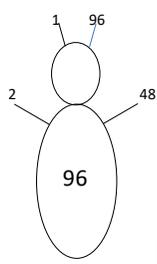
<https://diagnosticquestions.com>

10



FACTORS OF 96





Draw pairs of legs to represent all the other pairs of factors of 96. How many factors are there altogether?

Replace the ? marks.

```

    96
   / \
  2  48
   / \
  2  ?
   / \
 ?  ?
  / \
 ?  ?
 / \
 ?  ?
/  \
?  3
/  \
?  1
        
```

Complete this tree to show the prime factors of 96.

Explain this algorithmic method.

```

2 | 96
  | 48
  | 24
  | ?
  | ?
  | ?
  | ?
  | ?
  | 1
        
```

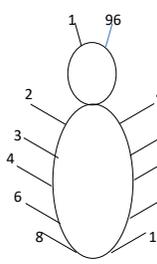
Write 96 as a product of prime factors.

11



FACTORS OF 96





96 has 12 factors: 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96

This is division by prime factors repeated until you get to 1.

```

    96
   / \
  2  48
   / \
  2  24
   / \
  2  12
   / \
  2  6
   / \
  2  3
   / \
  3  1
        
```

The prime factorization of 96 is $2^5 \times 3$

Explain this algorithmic method.

```

2 | 96
  | 48
  | 24
  | 12
  | 6
  | 3
  | 1
        
```

12



Highest Common Factor & Lowest Common Multiple



The prime factorization of 96 is $2^5 \times 3$
 The prime factorization of 30 is $2 \times 3 \times 5$

The LCM of 30 and 96 is _____?

The HCF of 30 and 96 is _____?

How do we use LCM and HCF when we add fractions? $\frac{1}{30} + \frac{1}{96}$



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Highest Common Factor & Lowest Common Multiple



The prime factorization of 96 is $2^5 \times 3$
 The prime factorization of 30 is $2 \times 3 \times 5$

The LCM of 30 and 96 is $2^5 \times 3 \times 5$

The HCF of 30 and 96 is 2×3

How do we use LCM and HCF when we add fractions?

$$\frac{1}{30} + \frac{1}{96}$$

$$= \frac{16+5}{2^5 \times 3 \times 5}$$

$$= \frac{21}{480}$$

$$= \frac{7}{160}$$


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HCF AND EQUIVALENT FRACTIONS

$0.5 = \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6} \dots \frac{2023}{4046} \dots \right\}$

You know about equivalence classes of fractions.
Many other other equivalence classes occur in mathematics

$0.2727 \dots = \left\{ \frac{3}{11}, \frac{6}{22}, \frac{9}{33} \dots \frac{81}{297} \dots \right\}$

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EASY CALC



$$\sqrt{\frac{1008}{847}}$$

What is the easiest way to do this calculation?

Did you spot that 7 is the HCF of 1008 and 847?

$$\sqrt{\frac{1008}{847}} = \sqrt{\frac{7 \times 144}{7 \times 121}}$$

$$= \sqrt{\frac{12^2}{11^2}}$$

$$= \frac{12}{11}$$

<https://aiminghigh.aimssec.ac.za/easy-calc/>

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**PATH TO THE STARS
PEOPLE MATHS
MULTIPLES
OF 3**

<https://aiminghigh.aimssec.ac.za/path-to-the-stars/>

17

STARS * MULTIPLES * CYCLES * CLOCK ARITHMETIC

Remainders when you divide by 7

0, 7, 14, 21...
1, 8, 15...
2, 9, 16...
3, 10, 17...
4, 11, 18...
5, 12, 19...
6, 13, 20...

Days of the week

Clock or Modular Arithmetic

Other examples include the 12-hour and 24-hour clocks and the months of the year. Numbering the points as you draw this 7-point star gives remainders when you divide any number in the list by 7.

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STARS * MULTIPLES * CLOCK ARITHMETIC

0, 5, 10 ...
4, 9, 14 ...
3, 8, 1 ...
2, 7, 12 ...
1, 6, 11 ...

0, 7, 14 ...
6, 13, 20 ...
5, 12, 19 ...
4, 11, 18 ...
3, 10, 17 ...
2, 9, 16 ...
1, 8, 15 ...

0, 7, 14 ...
6, 13, 20 ...
5, 12, 19 ...
4, 11, 18 ...
3, 10, 17 ...
2, 9, 16 ...
1, 8, 15 ...

0, 8, 16 ...
7, 15, 23 ...
6, 14, 22 ...
5, 13, 21 ...
4, 12, 20 ...
3, 11, 19 ...
2, 10, 18 ...
1, 9, 17 ...

MODULO 5 **MODULO 7** **MODULO 8**

Just as you cycle around the week, and Saturday comes every 7 days, the numbers in each list are EQUIVALENT in that modular system. Today is Saturday 11 February and 4th, 18th and 25th are also Saturdays! Numbers equivalent to 0 as you draw these stars are multiples of 5, 7 and 8.

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ROCK PAPER SCISSORS GAME WITH 7 ELEMENTS

**WHOLE CLASS GAME FOR 2 TEAMS
DIRECTED BY TEACHER**

Each player chooses an element and has an equal chance of winning or losing at each turn. Lines join each point to every other point. Arrows show which element wins.

Can you find 3, 4, 5, 6 and 7 cycles in this diagram?
e.g. Rock > Scissors > Sponge > Water > Rock is a 4-cycle.

<https://aiminghigh.aimssec.ac.za/years-scissors-paper-rock/>

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ROCK PAPER SCISSORS GAME WITH 7 ELEMENTS

Why play this game?
 Is the Rock, Paper, Scissors game a fair game?
 Is the 7-element game a fair game?

Cycles are a fundamental and important idea in Mathematics.
Can you spot 3, 4, 5, 6 and 7 cycles?
 3-cycle: Rock > Scissors > Paper > Rock
 4-cycle: Rock > Scissors > Sponge > Water > Rock
 5-cycle: Rock > Sponge > Paper > Air > Water > Rock
 6-cycle: Rock > Fire > Sponge > Paper > Air > Water > Rock
 7-cycle: Rock > Fire > Scissors > Sponge > Paper > Air > Water > Rock

<https://aiminghigh.aimssec.ac.za/years-scissors-paper-rock/>

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APPLICATIONS OF 11-CYCLES AND MODULAR ARITHMETIC

Modular arithmetic is used in cybersecurity to develop better, and more secure, encryption and error correcting codes.

Suppose you want to encrypt the number 1964 using the map $n \rightarrow 3n + 2$ as shown in the diagram.

In arithmetic modulo 11 the map $n \rightarrow 3n + 2$ is:
 $0 \rightarrow 2 \rightarrow 8 \rightarrow 4 \rightarrow 3 \rightarrow 0$ and
 $1 \rightarrow 5 \rightarrow 6 \rightarrow 9 \rightarrow 7 \rightarrow 1$ and
 $10 \rightarrow 10$
 so **1964** is encrypted as 5793.

To de-code 5793 use the mapping with the arrows reversed,
 so $5 \rightarrow 1$, $7 \rightarrow 9$, $9 \rightarrow 6$, $3 \rightarrow 4$

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ALGEBRAIC FACTORS & FRACTIONS

What is the HCF of $3(x^2 - 4)$ and $(2x^2 - x - 6)$?

What is the LCM of $3(x^2 - 4)$ and $(2x^2 - x - 6)$?

Simplify

$$\frac{1}{3(x^2-4)} + \frac{1}{2x^2-x-6}$$

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ALGEBRAIC FACTORS & FRACTIONS

What is the HCF of $3(x^2 - 4)$ and $(2x^2 - x - 6)$?

The factors of these expressions are:
 $3(x^2 - 4) = 3(x + 2)(x - 2)$
 $2x^2 - x - 6 = (2x + 3)(x - 2)$

The HCF of $3(x^2 - 4)$ and $(2x^2 - x - 6)$ is
 $(x - 2)$

What is the LCM of $3(x^2 - 4)$ and $(2x^2 - x - 6)$?

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ALGEBRAIC FACTORS & FRACTIONS



The LCM of $3(x + 2)(x - 2)$ and $(2x + 3)(x - 2)$
is $3(x + 2)(x - 2)(2x + 3)$

Now add the fractions

$$\frac{1}{3(x^2-4)} + \frac{1}{2x^2-x-6} = \frac{(2x+3)+3(x+2)}{3(x+2)(x-2)(2x+3)} = \frac{5x+9}{3(x+2)(x-2)(2x+3)}$$

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Factors of 360			
Powers of 2	Powers of 3	Powers of 5	Powers of 2 × powers of 3 × powers of 5
2 ⁰	3 ⁰	5 ⁰	1
		5 ¹	5
	3 ¹	5 ⁰	3
		5 ¹	15
2 ¹	3 ⁰	5 ⁰	2
		5 ¹	10
	3 ¹	5 ⁰	6
		5 ¹	30
2 ²	3 ⁰	5 ⁰	4
		5 ¹	20
	3 ¹	5 ⁰	12
		5 ¹	60
2 ³	3 ⁰	5 ⁰	8
		5 ¹	40
	3 ¹	5 ⁰	24
		5 ¹	120
	3 ²	5 ⁰	180
		5 ¹	360

TABLE TO RECORD ALL FACTORS OF 360

Finding the prime factorisation does not immediately give you all the factors including factors that are not prime.

For example, $360 = 2^3 \times 3^2 \times 5$.

Imagine a tree diagram presented in the table, with 4 branches for 2⁰ or 2 or 2² or 2³ then 4 sets of 3 branches for 3⁰ or 3 or 3² then 12 sets of 2 branches for 5⁰ or 5.

The table shows a systematic method for finding all the factors of 360 by listing all the numbers that are combinations of powers of 2⁰ or 2 or 2² or 2³ and 3⁰ or 3 or 3² and 5⁰ or 5.

Fill in the blanks.

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Factors of 360			
Powers of 2	Powers of 3	Powers of 5	Powers of 2 × powers of 3 × powers of 5
2 ⁰	3 ⁰	5 ⁰	1
		5 ¹	5
	3 ¹	5 ⁰	3
		5 ¹	15
2 ¹	3 ⁰	5 ⁰	2
		5 ¹	10
	3 ¹	5 ⁰	6
		5 ¹	30
2 ²	3 ⁰	5 ⁰	4
		5 ¹	20
	3 ¹	5 ⁰	12
		5 ¹	60
2 ³	3 ⁰	5 ⁰	8
		5 ¹	40
	3 ¹	5 ⁰	24
		5 ¹	120
	3 ²	5 ⁰	180
		5 ¹	360

TABLE TO RECORD ALL FACTORS OF 360

The table shows all powers of the prime factors of 360. Powers of 2 × powers of 3 × powers of 5 give the factors in the last column.

There are 4 different possible powers of 2
There are 3 different possible powers of 3
There are 2 different possible powers of 5
So, there are $4 \times 3 \times 2$ possible factors altogether.

There are 3 prime factors of 360 and 24 factors altogether.
 $360 = 2^3 \times 3^2 \times 5$

The factors written in order are
1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18
20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360

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THE BIG QUESTION IS: HOW MANY FACTORS?

When you know the prime factorisation of a number how do you find the **TOTAL** number of its factors?

Think of the tree structure for factors of 360 and then think of the same general idea for more factors and bigger factors.

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COUNT ALL THE FACTORS, NOT ONLY PRIME FACTORS AND 1.

For example, 20 has 6 factors 1, 2, 4, 5, 10 and 20.

1. How does the number 1 behave in the world of factors?
 2. What can you say about numbers that have exactly 2 factors?
 3. How many factors do square numbers have?
 4. How many factors do cube numbers have?
 5. Give other examples of numbers with 4 factors.
 6. Give examples of numbers with 5 factors.
 7. What about numbers with 6 factors?
 8. What do you notice about numbers with 2, 3, 4, 5 and 6 factors?
 9. What is the smallest number with exactly fourteen divisors?
 10. When you know the prime factorisation of a number, how do you find the TOTAL number of its factors?
 11. What is the smallest number with exactly 100 factors?
 12. Which number less than 1000 has the most factors?

Prime factors 2 and 5 highlighted

We'll take these questions one by one



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HOW MANY FACTORS?



1. How does the number 1 behave in the world of factors?

1 is a factor of every other number.
 It is neither prime, nor composite.

1 is the multiplicative identity,
 that is $1 \times n = n$ for every number n .

2. What can you say about numbers that have exactly 2 factors?

Prime numbers are numbers with exactly 2 factors, the number itself and 1. For example the factors of 7 are 1 and 7.

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HOW MANY FACTORS?



3. How many factors do square numbers have?

Numbers with exactly **3 factors** must be the square of a prime number.

For example $9 = 3^2$ has factors 1, 3 and 9.

Note that squares of composite numbers have more than 3 factors.

For example $14^2 = 2^2 \times 7^2 = 196$ has 9 factors
 1, 2, 4, 7, 14, 28, 49, 98, 196

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HOW MANY FACTORS



4. How many factors do cube numbers have?

Cubes of prime numbers have 4 factors,
 but cubes of composite numbers have more than 4 factors.

For example: $6^3 = 216$ has 16 factors:
 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 27, 36, 54, 72, 108, 216.

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HOW MANY FACTORS?



**5. Give other examples of numbers with 4 factors.
What do they have in common?**

Numbers with 4 factors are either the cube of a prime or the product of 2 primes.
For example $8 = 2^3$ has factors 1, 2, 4 and 8,
and 6 has factors 1, 2, 3 and 6.

In general, if a and b are prime numbers
then a^3 has 4 factors 1, a , a^2 , a^3
and ab has 4 factors 1, a , b , ab .

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HOW MANY FACTORS?



**6. Give some examples of numbers with 5 factors.
What do they have in common?**

The number $16 = 2^4$ has 5 factors 1, 2, 4, 8, 16.

All numbers with 5 factors are the 4th power of a prime.

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HOW MANY FACTORS



7. What about numbers with 6 factors?

The number $32 = 2^5$ has 6 factors 1, 2, 4, 8, 16, 32.
Also $12 = 2^2 \times 3$ has 6 factors 1, 2, 3, 4, 6, 12.

All numbers with 6 factors are
either the 5th power of a prime number or the
product of a square number and a prime number.

Notice that $30 = 2 \times 3 \times 5$ has 8 factors
1, 2, 3, 5, 6, 10, 15, 30

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HOW MANY FACTORS – RESULTS SO FAR – LOOK FOR PATTERNS



8. What do you notice about numbers with 2, 3, 4, 5 and 6 factors?

Numbers with exactly **2 factors** are prime,
For example the factors of 7 are 1 and 7.

Numbers with exactly **3 factors** are the square of a prime number.
For example the factors of $5^2 = 25$ are 1, 5 and 25.

Numbers with exactly **4 factors** are the cube of a prime number
or the product of two prime numbers.
For example $2^3 = 8$ has 4 factors: 1, 2, 4 and 8
and $2 \times 3 = 6$ has 4 factors 1, 2, 3 and 6

All numbers with **5 factors** are the 4th power of a prime.
For example $16 = 2^4$ has 5 factors 1, 2, 4, 8, 16.

All numbers with **6 factors** are either the 5th power of a prime number
or the product of a square number and a prime number.
For example 2^5 has factors 1, 2, 4, 8, 16, 32 and 4×5 has factors 1, 2, 4, 5, 10, 20

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THE BIG QUESTION IS: HOW MANY FACTORS?

Let's use what we have done so far to answer questions about numbers with more factors.

9. What is the smallest number with exactly fourteen factors?

Building on what we have done, n^6 has 7 factors, n^7 has 8 factors, n^8 has 9 factors... (for any prime number n).

Because 14 factorises into 7×2 , numbers with 14 factors are either of the form a^{13} or a^6b^1 (for primes a and b).

We choose a and b to be small to get the smallest number with 14 factors.

These are the 14 factors of 192:

$2^03^0 = 1$, $2^13^0 = 2$, $2^23^0 = 4$, $2^33^0 = 8$, $2^43^0 = 16$, $2^53^0 = 32$, $2^63^0 = 64$,
 $2^03^1 = 3$, $2^13^1 = 6$, $2^23^1 = 12$, $2^33^1 = 24$, $2^43^1 = 48$, $2^53^1 = 96$, $2^63^1 = 192$

So the smallest number with 14 factors is 192

Some other numbers with 14 factors:
 $2^{13} = 8192$ and $2^1 \times 3^6 = 1458$.



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HOW MANY FACTORS



10. When you know the prime factorisation of a number how do you find the TOTAL number of its factors?

For example, $12 = 2^2 \times 3$ has 6 factors 1, 2, 3, 4, 6, 12

$15 = 3 \times 5$ has 4 factors 1, 3, 5, 15

$32 = 2^5$ has 6 factors 1, 2, 4, 8, 16, 32.

$72 = 2^3 \times 3^2$ has 12 factors 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72

$96 = 2^5 \times 3$ has 12 factors 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96

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HOW MANY FACTORS? THE FUNDAMENTAL COUNTING PRINCIPLE



The fundamental counting principle is a rule used to count the total number of possible outcomes in a situation.

If there are m ways of doing something, and n ways of doing another thing after that, then there are $m \times n$ ways to perform both actions.

For the number of ways to perform 3 or more actions, multiply the number of ways of doing each action.

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HOW MANY FACTORS? THE FUNDAMENTAL COUNTING PRINCIPLE



10. When you know the prime factorisation of a number how do you find the TOTAL number of its factors?

A number with prime factorisation $a_1^{p_1} \times a_2^{p_2} \times a_3^{p_3} \times \dots \times a_r^{p_r} \times \dots \times a_n^{p_n}$ has prime factors a_r from $r = 1$ to $r = n$.

a_r contributes either a_r^0 or a_r^1 or a_r^2 or ... $a_r^{p_r}$ to the factors, altogether in $(p_r + 1)$ in different ways.

So a number with prime factorisation $a_1^{p_1} \times a_2^{p_2} \times a_3^{p_3} \times \dots \times a_r^{p_r} \times \dots \times a_n^{p_n}$ has $(p_1+1)(p_2+1)(p_3+1) \dots (p_n+1)$ factors.

$360 = 2^3 \times 3^2 \times 5$

2 contributes in 4 ways,
 3 contributes in 3 ways
 5 contributes in 2 ways
 to the factors of 360

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HOW MANY FACTORS? MORE QUESTIONS



- 11. What is the smallest number with exactly 100 factors?
- 12. Which number less than 1000 has the most factors?

These, and other similar questions, could be explored with paper and pencil using prime factorisation or it could be an opportunity for you to use a spreadsheet or simple coding (programming).

If you have an interest in coding you might wish to consider how to write a simple program to find all the factors of a number. For very large numbers, the realisation that you only need consider potential factors less than the square root of the number speeds up a program considerably!

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11. What is the smallest number with exactly 100 factors?

2^{99} is the smallest number with exactly 100 factors.
(the number will be bigger if it has factors 3, 5... or other primes)

12. Which number less than 1000 has the most factors?

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12. Which number less than 1000 has the most factors?

To find this number, and so you get a lot of factors, you have to try and multiply together the widest range of prime numbers starting with the lowest.

We try $2 \times 3 \times 5 \times 7 \times 11$ but the product > 1000 .
 $2 \times 3 \times 5 \times 7 < 1000$ so these are the numbers that we use.

$2 \times 3 \times 5 \times 7 = 210$ so we can use more powers of 2 to get more factors keeping the product under 1000.

The number closest to, and less than, 1000 that has the most factors is: $2^3 \times 3^1 \times 5^1 \times 7^1 = 840$

The number 840 has 32 factors.

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Multiples of 2

SCISSORS, PAPER, ROCK WITH 7 ACTIONS

HCF & LCM

$$\frac{1}{30} + \frac{1}{96} = \frac{16 + 5}{2^5 \times 3 \times 5} = \frac{21}{480} = \frac{7}{160}$$

THE FUNDAMENTAL COUNTING PRINCIPLE

How do you find the total number of factors without listing them?
For example: $15552 = 2^6 \times 3^5$ has $7 \times 6 = 42$ factors. Explain.

MULTIPLES, CYCLES & MODULAR ARITHMETIC

MODULO 5 **MODULO 7** **MODULO 8**

PATH TO THE STARS
PEOPLE MATHS
MULTIPLES OF 3

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A learning activity with links to:

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 - follow up ideas and links
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HOW MANY FACTORS AND RELATED RESOURCES

How Many Factors https://aiminghigh.aimssec.ac.za/how_many_factors/

Easy Calc <https://aiminghigh.aimssec.ac.za/easy-calc/>

Multiple Patterns <https://aiminghigh.aimssec.ac.za/multiple-patterns/>

Prime sieve <https://aiminghigh.aimssec.ac.za/prime-sieve/>

Path to the Stars <https://aiminghigh.aimssec.ac.za/path-to-the-stars/>

Scissors Paper Rock Game <https://aiminghigh.aimssec.ac.za/years-scissors-paper-rock/>

AIMSSEC YouTube Channel <https://www.youtube.com/@MathsToys>

GTEN website <https://gtenmaths.org/>

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Free lesson resources: <http://aiminghigh.aimssec.ac.za>

Collaborative Professional Development <https://aiminghigh.aimssec.ac.za/category/cpd>

AIMSSEC APP: <https://aimssec.app> (to download the resources and use resources offline)

AIMSSEC Website: <http://aimssec.ac.za>

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Thanks for participating.

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