Workshop for Teachers and Teaching Resources on the Proof of Pythagoras Theorem and its generalization to the Cosine Formula.

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Teaching Strategy: Developing Mathematical Reasoning and Proof

Curriculum content: Proof of Pythagoras Theorem by similarity. Proof of Cosine Formula.


The earliest record of the theorem is a Babylonian tablet about 3000 years old.

Prior knowledge: Understand that triangles that have the same angles are similar and the ratios of corresponding sides are equal. Similar triangles can be aligned so one is an enlargement of the other and the ratio of corresponding sides is the scale factor of the enlargement. Be able to calculate the areas of right angled triangles and squares.

Intended Learning Outcomes: At the end of this activity teachers and learners will:
- ✓ Know the logical difference between a theorem and its converse.
- ✓ Understand how to prove Pythagoras Theorem using similarity.
- ✓ Be able apply the theorem in problems in two and three dimensions.
- ✓ Appreciate that proof is essential to mathematics.
- ✓ Have experienced deriving formulas and developing proofs for themselves.

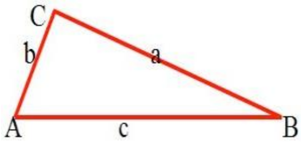
Fact box

Pythagoras Theorem:
If a triangle is right angled then the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides. If the sides have lengths *a*, *b* and *c* then $a^2+b^2 = c^2$.

Converse of Pythagoras Theorem:
If the area of the square on the longest side of a triangle is equal to the sum of the areas of the squares on the other two sides then the triangle is right angled.

Cosine Formula

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



$$\cos A = (b^2 + c^2 - a^2)/ 2cb$$

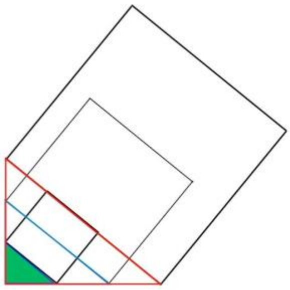
The Cosine Formula is a generalisation of Pythagoras Theorem
or alternatively Pythagoras Theorem is a special case of the Cosine Formula.

Workshop Activities for teachers

Activity 1 Warm-up 5 minutes

Organisation: Work in pairs. Every 10 minutes discuss your findings with the whole group.

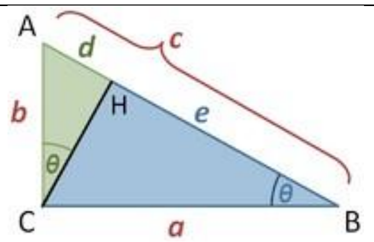
Resources: A board or flip chart for sharing ideas.



Discuss the similarities, enlargements and scale factors that you see in the diagram.
Note that the triangles are **not** isosceles triangles.
The perpendicular edges of the large triangle are split into three equal lengths.
Can the statements you make about this diagram be generalised to apply to enlargements for any scale factors?

Try this now

Activity 2 Proof of Pythagoras Theorem



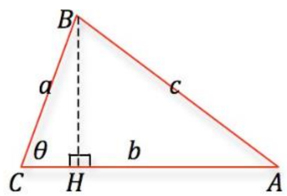
Angle ACB is a right angle and CH is perpendicular to AB.
Prove that ∠ACH=∠ABC, identify the similar triangles in this diagram and prove that they are similar.
Use similarity to give d in terms of b and c
and then to give e in terms of a and c.
What can you deduce from these results?

Try this now

Notes to help you do the activity.

1. Don't spend long on Activity 1. It is just to get you thinking about similarity and enlargement.
2. By asking questions about the diagram in Activity 2 teachers can guide learners to derive the proof of Pythagoras Theorem by similar triangles for themselves.
3. By asking questions in Activity 3 teachers can guide learners to derive the proof of the Cosine Formula for themselves.
4. You need to think clearly about the difference between a theorem 'we know P is true so it follows that Q is true' (given for some statements P and Q) and its converse 'we know Q is true so it follows that P is true'. In the case of Pythagoras Theorem both theorem and converse are true but with other theorems this is not always the case.
5. By asking questions in Activity 4 teachers can guide learners to derive the proof of the Converse of Pythagoras Theorem for themselves.

Activity 3 Cosine Formula

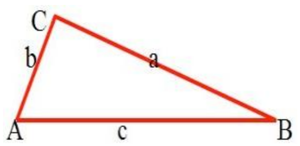


$c^2 = a^2 + b^2 - 2ab\cos\angle ACB$

In any triangle ABC, let H be the foot of the perpendicular from B to AC. Now think about the right angled triangle BHC.
Show BH = a sin θ, CH = a cos θ and AH = b – a cos θ.
Now use Pythagoras Theorem for triangle AHB to prove the Cosine Formula for triangle ABC.
What does the cosine formula become when **∠ACB =90°**?
Discuss how the cosine formula gives a generalisation of Pythagoras Theorem.

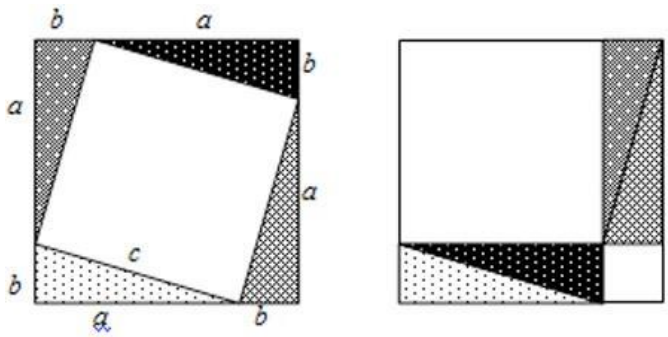
Try this now

Activity 4 Converse of Pythagoras Theorem



You have proved the cosine formula for any triangle ABC, that is:
$c^2 = a^2 + b^2 - 2ab\cos \angle ACB$
Suppose you don't know ∠ACB but you do know that $\mathbf{c^2 = a^2 + b^2}$.
In this case what does the cosine formula tell you about ∠ACB?
This is the converse of Pythagoras Theorem. Try this now

Discussion of pedagogical issues:



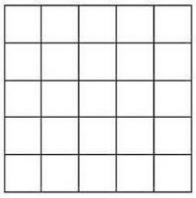
1. Discovery and proof are essential elements in learning mathematics.
2. When teachers first introduce Pythagoras Theorem, they often use the diagram shown on the right to give learners the opportunity to use their knowledge of areas to discover the theorem for themselves.
3. Four **identical (congruent) right angled triangles** are arranged in the squares shown in the diagrams which have sides (a + b). Think about the areas in the diagrams. Just take away the four identical right angled triangles from each of the diagrams. What is left? What does this tell you about the areas of the unshaded parts of the diagrams? Write this as a formula involving a, b and c.
4. To introduce Pythagoras Theorem by telling learners the formula is not good teaching. It robs learners of the opportunity to think mathematically and to understand where the formula comes from. Mathematics should always be taught so that learners understand how to derive the formulas, and why the methods work, even if the proofs are not assessed by examination.
5. Always be on the lookout for connections. It is important to recognise that the Cosine Formula is a generalisation of Pythagoras Theorem.
6. The proof of the converse of Pythagoras Theorem (Activity 4) follows simply from the proof of the Cosine Formula.

References: See the interactivity Matter of Scale http://nrich.maths.org/811 and http://nrich.maths.org/6553 . For more proofs of Pythagoras Theorem, the wonderful website Cut the Knot has 88 versions http://www.cut-the-knot.org/pythagoras/

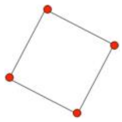
Classroom Activities for Learners

Resources: Scrap paper, ruler, scissors, calculators.

Activity 1 Squares Game - warm up 5 minutes, can be used often

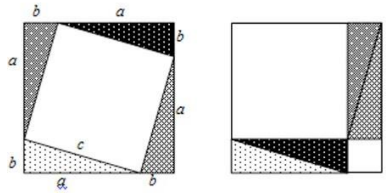


How to play the game: This is a game for 2 players or two teams. Take it in turn to mark one intersection point with your colour (or special mark) on a grid like the one on the left. Try to make a square with your own marks at the four corners, Try to stop your opponent making a square. Don't forget the tilted squares. The winner is the first player to make a square.



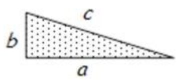
Activity 2 Investigation of Squares Built on the Sides of a Right Angled Triangle 1 hour

Look at the two diagrams on the right. All the triangles are congruent. If you take away the 8 copies of the triangle, what is left behind? Think about areas? Discuss what you see and what you think it means.



To investigate this further and check all the ideas, cut out 4 identical right angled triangles. Each learner in the class will probably have a different triangle.

1. Measure the three sides *a, b* and *c* in centimetres correct to the nearest millimetre (e.g. 5.2 cm) and write down your measurements.

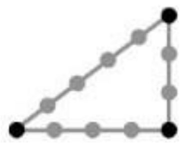


2. Arrange your triangles like the diagram on the left above. What shape is the quadrilateral in the middle surrounded by the four triangles? How can you be sure?
3. What is its area in terms of *c*? Calculate the area for your own triangle.
4. From your measurements add up $(a+b)$. Draw a square with sides of length $(a + b)$. Arrange your 4 triangles in the square you have drawn like the diagram on the right.
5. How can you be sure that the two uncovered (unshaded) quadrilaterals are squares?
6. What are the areas of these two squares in terms of *a* and *b*? Calculate these two areas.
7. Explain why the diagrams show that $a^2+b^2=c^2$. Do your calculations agree with this?
8. What have you discovered about the areas of squares built on the sides of your right angled triangle?
9. Does this always work for all values of *a* and *b*? Why?

Notes

Ideally learners will have met the proof of Pythagoras Theorem given in Activity 2 in lower secondary school but it is included here for use where this is not the case or for review purposes.

Activity 3 Making right angles. The Converse of Pythagoras Theorem

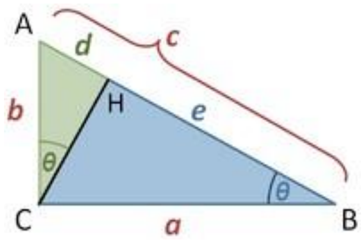


Resources needed: Rope or string 4 metres long knotted at intervals of 10 centimetres. Tie knots in your rope before the lesson. It will be useful for other work on measurement.

a. Get three learners to hold the rope to make a triangle with sides of length 3 units, 4 units and 5 units. Ask the class what they notice. Everyone should check that $3^2 + 4^2 = 5^2$ so that this triangle must be a right angled triangle. Repeat the same activity for a triangle with sides of length 6 units, 8 units and 10 units. Ask the learners what connection this triangle has with the 3-4-5 triangle. Repeat the same activity for a triangle with sides of length 5 units, 12 units and 13 units.
b. Discuss with the class the difference between saying "If the triangle is right angled then $a^2+b^2=c^2$" and " If $a^2+b^2=c^2$ then the triangle is right angled". Ask the class "What do you know in each case and what follows?" Get the class to make up some "If... then..." sentences of their own where "If P then Q" is true but "If Q then P" is false.

Classroom Activities for Learners

Activity 4 Proof of Pythagoras Theorem by Similarity **Time:** 1 hour

What the teacher does	**What the learners do**
Draws the diagram on the board before the lesson. Does **not** tell the learners that this is a lesson on Pythagoras Theorem. Goes around the classroom checking the drawings that learners are making and helping them to make the drawing correctly.	Angle ACB is a right angle. CH is perpendicular to AB.  Learners copy the diagram carefully in pencil, erasing and making corrections if necessary.
Asks learners to identify the similar triangles in the diagram. Gives time for learners to discuss this in pairs. Checks that learners have identified 3 triangles all similar to each other.	Work independently, and then in pairs, to identify similar triangles. Record their results.
Asks: "Can you use similarity to give d in terms of b and c?" Goes around the class checking that learners understand and can succeed. Asks: "Can you use similarity to give e in terms of a and c?" Goes around the class checking that learners can succeed.	Work independently to find d = b^2/c and then to find e = a^2/c. Record their results.
Asks "What do d and e add up to? What conclusion can you make?" Gives time for learners to work independently, then to discuss in pairs. Facilitates a class discussion. Expects learners to make the connection with Pythagoras Theorem.	Discuss in pairs what they get from d + e and the significance of this result. Record their results. Join in a class discussion. Record their conclusions.

Ideas for teaching:
Don't rush the process of finding the proofs in these lessons. Don't tell your learners what to do. Give time for the learners to work out the proofs for themselves or in discussion with a partner. Good teaching results in learners developing problem solving and communication skills and independence. If your learners are dependent on you, and they expect you to tell them what to do, then ask yourself if you need to change your style of teaching.

The proof of the Cosine Formula as in Activity 3 in the Teacher Workshop can be introduced in a similar way as a separate lesson, giving the learners the diagram, asking them leading questions and allowing them to find the proof for themselves. The teacher should ask the class to discuss the connection between the Cosine Formula and Pythagoras Theorem. At this stage the proof of the Converse of Pythagoras Theorem can be introduced.

Make sure that all the learners understand the proofs of these results then give them some problems for practice in applying the results. If you have access to a computer then use the many good resources available such as Matter of Scale nrich.maths.org/811 and the resources at www.cut-the-knot.org/pythagoras/

Discussion of Teaching Strategy – Developing Mathematical Reasoning and Proof

Rather than starting the lesson with a statement or proof of the theorem teachers should guide learners by asking questions about the activities, encouraging learners to be active and curious, to spot patterns, to create their own conjectures, to ask their own questions, to experiment with ideas, to help one another, to set goals for themselves, to monitor their own progress and to know that making mistakes and 'being stuck' are part of learning. Later in the lesson teachers should base their explanations of the reasoning behind the proofs of the theorem on the learners' own experiences.

Games can be used to encourage new ways of thinking and to engage learners in exploring ideas and pathways of reasoning new to them. According to (Scarfe 1962) "All play is associated with intense thought activity and rapid intellectual growth. The highest form of research is essentially play. Einstein is quoted as saying: the desire to arrive finally at logically connected concepts is the emotional basis of a vague play with basic ideas. This combinatory or associative play seems to be the essential feature in productive thought."

As a lesson starter, the Squares Game can lead to Pythagoras Theorem or coordinates. It reinforces recognition of perpendicular directions and you can make the connection with ideas of gradient (distance up divided by distance across). As a class game it gives teachers the opportunity to act as judge when a square is claimed in error, or when a player fails to recognise a square, and to explain how to recognise a square by looking at corresponding distances across and up, the lengths of edges, the right angles and the symmetry.

The activities in this chapter provide scaffolding for learners to develop different proofs of Pythagoras Theorem. In addition the activities bring in the important ideas of generalizing a theorem, in this case the proof of the cosine rule as a generalization of Pythagoras Theorem, and also of proving both a theorem and its converse. To help learners to appreciate the need to prove the converse, teachers should introduce some statements of the form 'If p then q' where the converse 'if q then p' is false and the class should discuss these logical forms and suggest some examples of their own. Examples like the following are easy to invent:

i) 'if it is a fish then it is not a bird' (true) and 'if it is not a bird then it is a fish' (false)

ii) 'if $x = 7$ then $x > 5$' (true) and its converse: 'if $x>5$ then $x = 7$' (false)

iii) 'if he is my brother then I am his sister' (true) and if I am his sister then he is my brother (also true).

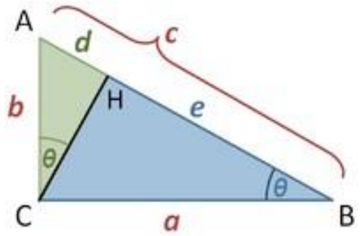
By playing with four copies of a right angled triangle, and rearranging them in different configurations as in Classroom Activity 2, learners can explore ideas of what stays the same and what changes between the two arrangements and discover for themselves facts about the areas of the squares on the sides of right angled triangles. As the same line of reasoning applies to any right angled triangle the learners will have discovered a proof of Pythagoras Theorem for themselves.

Learners can also be guided from step to step by questioning about similar triangles (as in the Key Questions below) to prove the theorem using similar triangles and then to generalise the proof to proving the cosine formula.

1962 November, Childhood Education, Volume 39, Issue 3, "Play is Education" by N. V. Scarfe, Start Page 117, Quote Page 120, Published Association for Childhood Education International, Washington D.C. (Verified with scans; thanks to Jacksonville, Florida public library)

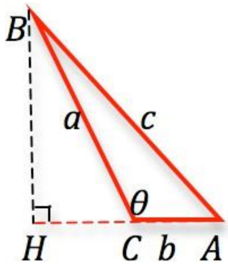
Key Questions

Can you answer these questions? Can you give explanations for all your answers?

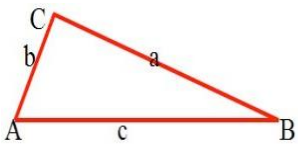


1. In this triangle angle ACB is a right angle and H is the foot of the perpendicular from C to AB.

Which triangles are right angled?
How would you prove that ∠ACH = ∠ABC?
Which triangles are similar and why?
Explain the reasons that you can write d in terms of b and c.
Explain the reasons that you can write e in terms of a and c.
Explain how these results lead to a proof of Pythagoras Theorem.

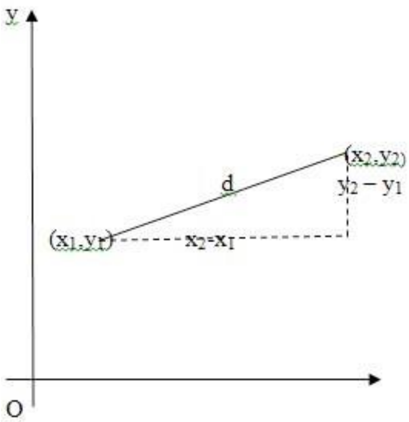


2. How would you change the proof of the Cosine Formula for the case where ∠ACB is obtuse?



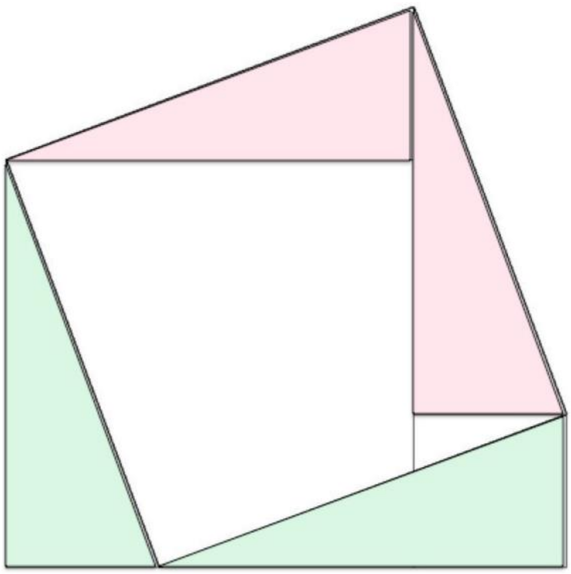
3. In triangle ABC, what can you say about ∠ACB in the three cases:
   i. $c^2 < a^2 + b^2$
   ii. $c^2 = a^2 + b^2$
   iii. $c^2 > a^2 + b^2$

4. Make up 3 statements where the statement is true and the converse is false.

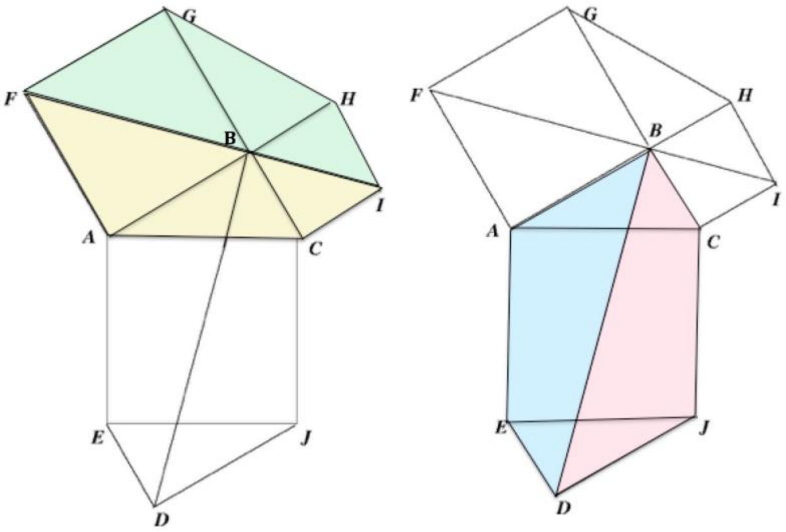


5. Explain how the formula for the distance between two points in the Cartesian plane is simply another way of stating Pythagoras Theorem.

Make Posters
Give different groups of learners the task of making posters to explain different proofs of Pythagoras Theorem. The learners can then be given the task to explain their proof to the class, either using their poster or using a powerpoint presentation. The posters can then be put up on the classroom wall to help learners to remember the reasoning behind the proofs.



Perigal's Dissection
This diagram provides a proof without words of Pythagoras Theorem.
Can you see it?



Leonardo da Vinci's Proof
The diagrams are drawn with squares on the sides of the right angled triangle ABC. Can you spot the two congruent copies of ΔABC?

Can you prove that the four quadrilaterals FGHI, FICA, ABDE and BCJD are all congruent and therefore equal in area?

Now think about the coloured hexagons that are each made up of two congruent quadrilaterals. They have equal areas.

Remove two copies of the right angled triangle from the hexagon FGHICA and you are left with the squares on the two shorter edges of the right angled triangle.

Remove two copies of the right angled triangle from the hexagon ABCJDE and you are left with the square on the hypotenuse of the right angled triangle.

Perigal's Proof.
Remove two copies of the triangle and you have the square on the hypotenuse.

Remove the other two copies of the triangle and you have the squares on the other two sides.