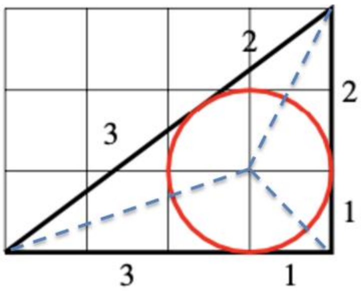
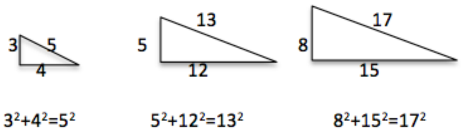
PYTHAGOREAN TRIPLES AND INCIRCLES

Investigate inscribed circles of right-angled triangles that have edges of integer length.



(1) Draw a 3-4-5 right-angled triangle on a grid. Using compasses construct the internal bisectors of the angles of the triangle and then draw the inscribed circle (we'll call it an incircle). What do you notice?



(2) Repeat this for a 5-12-13 right-angled triangle and then for an 8-15-17 right-angled triangle. Does the radius of the incircle have integer length for the triangles given by the three Pythagorean triples 3-4-5, 5-12-13 and 8-15-17?

(3) To prove the general result that the radius of the incircle is an integer for all right-angled triangles with edges of integer length you will need to use algebra and the formulae for generating Pythagorean Triples.

The formula is: for any two whole numbers p and q then the triangle with integer edge lengths $p^2 + q^2$, $p^2 - q^2$ and $2pq$ is right angled. Check using Pythagoras Theorem.

(4)Check that, $p = 2$ and $q = 1$ gives the 3–4–5 triangle,

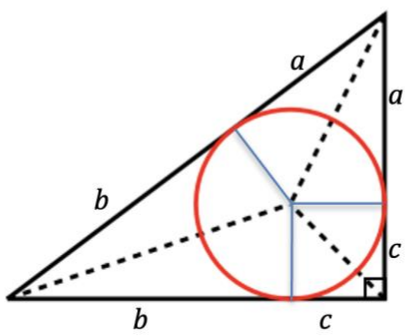
$p = 3$ and $q = 2$ gives the 5–12–13 triangle and,

$p = 4$ and $q = 1$ gives the 8–10–17 triangle.

What triple do you get by taking $p = 4$ and $q = 3$?

(3, 4, 5)	(5, 12, 13)	(8, 15, 17)	(7, 24, 25)
(20, 21, 29)	(12, 35, 37)	(9, 40, 41)	(28, 45, 53)
(11, 60, 61)	(16, 63, 65)	(33, 56, 65)	(48, 55, 73)
(13, 84, 85)	(36, 77, 85)	(39, 80, 89)	(65, 72, 97)

See Wikipedia for more information. This is a list (in a different notation) of all the Pythagorean Triples for triangles with edge lengths less than 100, not listing enlargements of these triangles.



(5) **Theorem** For all right-angled triangles with integer edge lengths, the radius of the incircle is an integer.

To prove this Theorem:

(i)Write down three simultaneous equations giving $a + b$, $b + c$ and $c + a$ in terms of p, q and r.

(ii)Solve the equations to give a, b and c in terms of p, q and r.

(iii) Prove that the radius of the incircle is equal to c and $c = q(p - q)$ which is an integer.

HELP

To solve the three simultaneous equations for the proof in (5) just use the methods you know for solving two simultaneous equations. These are easy equations to solve because the coefficients are all 1.

NEXT

Prove that if a right-angled triangle with edges of integer lengths has an inscribed circle whose radius is 1, then the triangle is a 3-4-5 right-angled triangle.