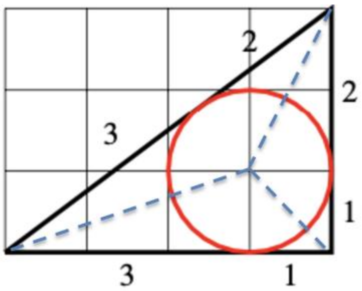
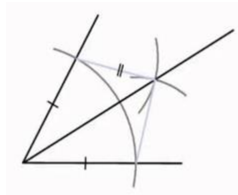
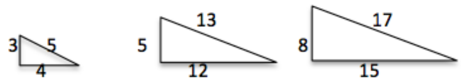
PYTHAGOREAN TRIPLES AND INCIRCLES

Investigate inscribed circles of right-angled triangles that have edges of integer length.



(1) Draw a 3-4-5 right-angled triangle on a grid. Using compasses construct the internal bisectors of the angles of the triangle and then draw the inscribed circle (we'll call it an incircle). What do you notice?





$3^2+4^2=5^2$ $5^2+12^2=13^2$ $8^2+15^2=17^2$

(2) Repeat this for a 5-12-13 right-angled triangle and then for an 8-15-17 right-angled triangle. Does the radius of the incircle have integer length for the triangles given by the three Pythagorean triples 3-4-5, 5-12-13 and 8-15-17?

(3) To prove the general result that the radius of the incircle is an integer for all right-angled triangles with edges of integer length you will need to use algebra and the formulae for generating Pythagorean Triples.

The formula is: for any two whole numbers p and q then the triangle with integer edge lengths p^2+q^2, p^2-q^2 and $2pq$ is right angled. Check using Pythagoras Theorem.

(4)Check that, $p = 2$ and $q = 1$ gives the 3–4–5 triangle,

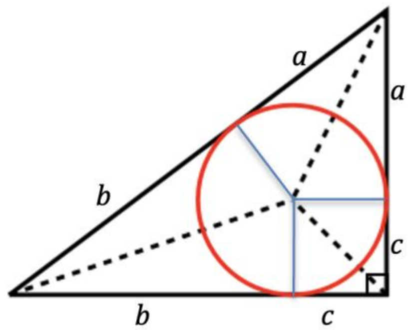
$p = 3$ and $q = 2$ gives the 5–12–13 triangle and,

$p = 4$ and $q = 1$ gives the 8–10–17 triangle.

What triple do you get by taking $p = 4$ and $q = 3$?

(3, 4, 5)	(5, 12, 13)	(8, 15, 17)	(7, 24, 25)
(20, 21, 29)	(12, 35, 37)	(9, 40, 41)	(28, 45, 53)
(11, 60, 61)	(16, 63, 65)	(33, 56, 65)	(48, 55, 73)
(13, 84, 85)	(36, 77, 85)	(39, 80, 89)	(65, 72, 97)

See Wikipedia for more information. This is a list (in a different notation) of all the Pythagorean Triples for triangles with edge lengths less than 100, not listing enlargements of these triangles.



(5) **Theorem** For all right-angled triangles with integer edge lengths, the radius of the incircle is an integer.

To prove this Theorem:

(i)Write down three simultaneous equations giving $a + b$, $b + c$ and $c + a$ in terms of p, q and r.

(ii)Solve the equations to give a, b and c in terms of p, q and r.

(iii) Prove that the radius of the incircle is equal to c and $c = q(p - q)$ which is an integer.

HELP

To solve the three simultaneous equations for the proof in (5) just use the methods you know for solving two simultaneous equations. These are easy equations to solve because the coefficients are all 1.

NEXT

Prove that if a right-angled triangle with edges of integer lengths has an inscribed circle whose radius is 1, then the triangle is a 3-4-5 right-angled triangle.

NOTES FOR TEACHERS

Why do this activity?

This is an enrichment activity for older students calling for them to extend their ideas of solving simultaneous equations from two variables to three. It also provides a glimpse of deeper ideas of parametric formulas and Diophantine equations that are studied at undergraduate level.

It's important for students to understand the difference between a theorem and its converse and that one may be true and the other false. The use of Pythagorean triples in surveying in ancient civilisations provides a real-life application of the converse of Pythagoras Theorem.

Perspective about what they learn in school, as well as appreciation of different cultures, is gained from reference to the history of the study of 'Pythagorean' ideas in Ancient Babylon (1800 BCE) and Ancient India (800 BCE) long before Pythagoras who lived from 570–495 BCE.

Learning objectives

At the end of this activity learners will:

- have revised Pythagoras Theorem and its converse;
- know the logical difference between a theorem and its converse;
- appreciate the connectedness of algebra and geometry;
- understand the geometrical facts underlying the method for the construction of the inscribed circle in a triangle;
- appreciate that proof is essential to mathematics;
- have solved simultaneous equations in three variables.

Suggestions for teaching

Lower Secondary

Statement (or theorem) and its converse.

To help learners to appreciate the need to prove both theorem and converse, teachers should introduce some statements of the form 'If p then q' where the converse 'if q then p' is false and the class should discuss these logical forms and suggest some examples of their own. Examples like the following are easy to invent:

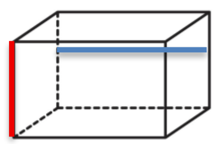
(i) 'If it is a fish then it is not a bird' (true) and 'if it is not a bird then it is a fish' (false).

(ii)'If x = 7 then x > 5' (true) and its converse: 'if x>5 then x = 7' (false).

(iii)'If he is my brother then I am his sister' (true) and if I am his sister then he is my brother (also true).

(iv) Statement: 'If a quadrilateral is a square, then it has four equal sides' (true).
Converse: 'If a quadrilateral has four equal sides, then it is a square' (false, it could be a rhombus).

(v) Statement: 'If two lines are parallel, then they do not meet' (true).
Converse: If two lines do not meet, then they are parallel' (false, they could be skew lines in 3D, e.g. the red and blue lines in the diagram).



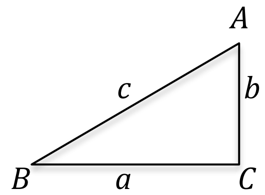
(vi) Statement: 'If a number is divisible by 4, then it is even' (true).
Converse: 'If a number is even, then it is divisible by 4' (false, e.g. 6 and 10).

Next ask learners to state the converse of Pythagoras Theorem. Then explain that there are many proofs and you have chosen a proof that only uses the fact that if lengths of the corresponding edges in two triangles are equal then the triangles are congruent (SSS). This proof works well for students because it only uses Pythagoras' theorem, triangle congruence (SSS) and no trigonometry or coordinate geometry. It is very close in spirit to surveying and construction methods.

Converse of Pythagoras' Theorem

If a triangle has edges of lengths a, b, and c (with c the longest side) and $a^2 + b^2 = c^2$, then the triangle is right-angled, with the right angle opposite the side of length c.

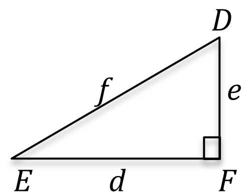
Proof



1. Start with a triangle ABC with the given side lengths

$AB = c$, $BC = a, CA = b$, where c is the longest side and $a^2 + b^2 = c^2$.

2. Construct a right-angled triangle DEF for comparison such that:
   - $EF = d = a$,
   - $FD = e = b$,
   - the right angle is at F.



By Pythagoras' Theorem, $d^2 + e^2 = f^2$ so $DE = f = c$.

3. Compare the two triangles:
   Triangle ABC and triangle DEF now have:
   - one side of length a,
   - one side of length b,
   - one side of length c.

   Therefore, the two triangles are congruent (SSS congruence).

4. Since triangle DEF is right-angled at E, and the triangles are congruent, the angle in triangle ABC opposite the side of length c must also be a right angle.

Conclusion

Triangle ABC is right-angled.

If $a^2 + b^2 = c^2$, then the triangle is right-angled.

This proves the converse of Pythagoras' Theorem.

The earliest record of the study of Pythagorean triples dates to around 1800–1600 BCE, in ancient Mesopotamia (Babylonia). This is roughly 1,200 years before Pythagoras (who lived c. 570–495 BCE).

The Plimpton 322 Babylonian clay tablet comes from Southern Mesopotamia (modern Iraq) and is dated ~1800 BCE. It shows, in cuneiform writing, a table of integer solutions to $a^2 + b^2 = c^2$, that is Pythagorean triples. For example (119, 120, 169), (4601, 4800, 6649), and others. It shows the Babylonians systematically generated and used Pythagorean triples, likely for surveying and construction. Without protractors and without measuring angles they could produce accurate right angles simply by measuring distance.

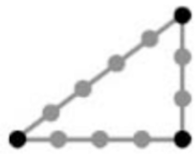
Similarly, in ancient Egypt, “rope-stretchers” used knotted ropes with equal spacing to form right angles. In Roman surveying, right angles were set out using measured lengths. In modern construction, the 3–4–5 check is still used to verify squareness, even with laser tools.

Making right angles. The Converse of Pythagoras Theorem

Resources needed: *Rope or string 4 metres long knotted at intervals of 10 centimetres. Tie knots in your rope before the lesson. It will be useful for other work on measurement.*

(a) Both Pythagoras Theorem and its converse are true. Discuss with the class the difference between saying “If the triangle with edges *a, b* and *c* is right angled then $a^2 + b^2 = c^2$” and “If $a^2 + b^2 = c^2$” then the triangle is right angled”. Ask the class “What do you know in each case and what follows?”

(b) Get three learners to hold the rope to make a triangle with sides of length 3 units, 4 units and 5 units. Ask the class what they notice. Everyone should check that $3^2 + 4^2 = 5^2$ so that this triangle must be a right-angled triangle.



(c) Repeat the same activity for a triangle with sides of length 6 units, 8 units and 10 units. Ask the learners what connection this triangle has with the 3-4-5 triangle (an enlargement of scale factor 2 and also a right-angled triangle).

(d) Repeat the same activity for a triangle with edges of length 5 units, 12 units and 13 units.

Upper Secondary

If they have not done it before go through the lesson described above for Lower Secondary.

Give the learners the worksheet on the first page . Get them to work in pairs and help each other if needed. For each of the five sections ask learners to come to the front of the class and explain how they worked out that section, but draw the diagrams for them on the board beforehand.

Key questions

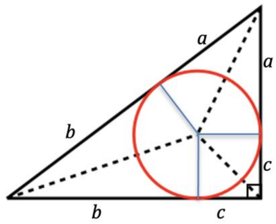
- Why does the intersection of the bisectors of the angles of a triangle give the centre of the inscribed circle?
- Can you prove that there is a square in the diagram which shows that the radius of the incircle is *c*?

SOLUTIONS

Theorem For all right-angled triangles with integer edge lengths, the radius of the incircle is an integer.

Proof

(i)Write down three simultaneous equations giving $a + b$, $b + c$ and $c + a$ in terms of p, q and r.



$a + b = p^2 + q^2$ (1)

$b + c = p^2 - q^2$ (2)

$c + a = 2pq$ (3)

(ii) and (iii) Solve the equations to give a, b and c in terms of p, q and r and prove that the radius of the incircle is equal to c and $c = q(p - q)$ which is an integer.

$a - c = 2q^2$ (4) Subtracting (2) from (1)

$2a = 2pq + 2q^2$ Adding (3) and (4)

$a = q(p + q)$

$b = p(p - q)$

$c = q(p - q)$

As p and q are integers, the radius of the incircle c is an integer.

NEXT

Theorem If a right-angled triangle with edges of integer lengths has an inscribed circle whose radius is 1, then the triangle is a 3-4-5 right-angled triangle.

If $c = 1$ then $(a + 1)^2 + (b + 1)^2 = (a + b)^2$.

This gives $a^2 + 2a + 1 + b^2 + 2b + 1 = a^2 + 2ab + b^2$ so that

$$a + b + 1 = ab$$

That is $(a - 1)(b - 1) = 2$. The only factors of 2 are 1 and 2 so either

$a - 1 = 2$ and $b - 1 = 1$ or $a - 1 = 1$ and $b - 1 = 2$.

In both cases the triangle is a 3-4-5 right angled triangle.

Follow up

Click here to download a pdf that provides both a framework and activities for workshops for teachers and teaching resources for learners on Pythagoras Theorem and its generalisation to the Cosine Rule.

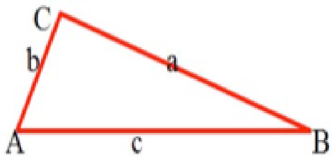
Cosine Formula

The Cosine Formula is a generalisation of Pythagoras Theorem
or alternatively Pythagoras Theorem is a special case of the Cosine Formula.

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = c^2 + a^2 - 2ca\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$



$\cos A = (b^2 + c^2 - a^2)/\ 2cb$

Go to the **AIMSSEC AIMING HIGH** website for lesson ideas, solutions and curriculum links: http://aiminghigh.aimssec.ac.za

Subscribe to the **MATHS TOYS YouTube Channel** https://www.youtube.com/c/MathsToys/videos

Download the whole AIMSSEC collection of resources to use offline with the **AIMSSEC App** see https://aimssec.app or find it on Google Play.

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and school years up to Secondary 5 in East Africa.
New material will be added for Secondary 6.
For resources for teaching A level mathematics (Years 12 and 13) see https://nrich.maths.org/12339
Mathematics taught in Year 13 (UK) & Secondary 6 (East Africa) is beyond the SA CAPS curriculum for Grade 12

	Lower Primary Approx. Age 5 to 8	Upper Primary Age 8 to 11	Lower Secondary Age 11 to 15	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13