

The **FIFTEEN GAME COLLECTION Inclusion and Home Learning Guide** is part of a Learning Pack downloadable from the AIMING HIGH website <https://aiminghigh.aimssec.ac.za> on the common theme of **MATHEMATICAL DISGUISES or ISOMORPHISM**. It includes the related games of Pat the Pig, Siege, Magic Squares and Ultimate Tic Tac Toe, together with guidance for supporting the learning, for inclusion in school lessons and home learning for all ages and learning stages from pre-school to school-leaving.

Choose what seems suitable for the age or attainment level of your learners.

Colour coding for age groups or school phases	Early Years and Lower Primary	Upper Primary	Lower Secondary	Upper Secondary
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The Fifteen Game, Magic Square Game, Siege, Pat the Pig, and Noughts and Crosses (Tic Tac Toe) are all essentially the same game set in different contexts. All the games can be played in a single day or they can be introduced at different times as seemingly unrelated games. If students are encouraged to look for, and discuss, similarities they will spot that all these games have corresponding features. For younger students the concept of the games being in different ‘Mathematical Disguises’ could be introduced as a simpler name for the correct mathematical term - Isomorphism (Greek iso-same and morph-change). Isomorphism is tremendously important in mathematics. It carries with it the crucial information about the object while discarding irrelevant information.

In all these games the relationships and structure are essentially the same but the contexts are different. To some extent these contexts are irrelevant. If you recognise the structure and have seen it in another form, you already know a lot about it, and that saves work, thus isomorphism is very useful. The correspondences between the games can be discovered by students if they play all the games.

Why play these games in lessons?

The FIFTEEN and MAGIC SQUARE games give learners a lot of practice in using number bonds and, in so doing, they build ‘number sense’ and numeracy. All the games develop logical thinking because the incentive to win leads to concentration and thinking several moves ahead. To win, players need to ask themselves “If my opponent made that move, how would it affect my play?” ... “What if...?”, Then what...?”

Learning objectives

In doing this activity students will have an opportunity to:

- practise addition;
- think strategically;
- make connections between different mathematical ideas.

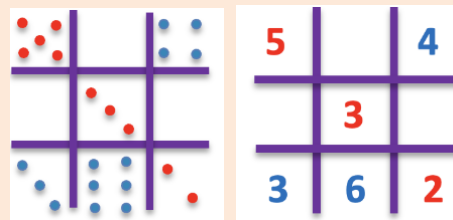
Generic competences

- **think mathematically** and give explanations to develop communication skills;
- **visualize** and interpret images to represent concepts and situations;
- **work systematically** to investigate all possible cases;
- **develop the life skill of** winning and losing with consideration for others.

Early Years and Lower Primary

MAKE TEN GAME

This game for two players is suitable for very young players, and fun for everyone. You need one die and a coloured pen for each player. The game helps learners to learn the number bonds for 10 and reinforces recall of those bonds, and it is a good preparation for playing the Fifteen Game. Young players can play the game by copying the spots that appear on the die, drawing the spots in the squares and counting the spots. Later they can play it by writing numbers instead of drawing spots.



Draw a frame with four lines as for Noughts and Crosses (Tic, Tac, Toe). Until all the squares are filled in, take turns to throw the die and draw dots in one of the squares to record the number that comes up on the die, or write the number. Players aim to make a line of three numbers adding up to 10, horizontally vertically or diagonally. They score 2 points if the spots or numbers are all in their own colour and 1 point if they are in different colours. In the game shown, Red has scored two points for 5, 3 and 2 in a diagonal line and Blue has scored 1 point for 3, 3 and 4 in line diagonally. At the next throw a 1, 2 or 4 will score.

The whole class can be split into 2 teams to play the game. The teacher draws a big grid on the board. Each team has a team leader who throws the die for the team. Then the team members discuss where to put the number and, when they come to a decision, the team leader tells the teacher where to put it on the board.

When introducing to the Fifteen Game make a set of cards for 1 to 9 with **spots on instead of numbers**. Play the Fifteen Game using these cards so that little children can count the dots. When the children can recognise the written numbers, use number cards. Some 7 and 8-year-olds will be able to write down the winning combinations of 3 numbers that add up to 15. Don't rush this. If they don't find all 8 combinations one day then encourage them to keep looking and come back to it another day. Eventually they will find all 8 combinations. Help the learners to arrange the list systematically so that they can check that they have found all the combinations.

FIFTEEN GAME for 2 players or 2 teams



Nine numbered cards or just pencil and paper.



Take turns to choose one of the numbers 1 to 9 until all the numbers are taken.

It is now your number and your opponent cannot choose it. Each number can only be chosen once.

To win, be the first to pick 3 numbers that add up to 15.

HELP

- Write down all the combinations of three of the numbers that add up to 15, for example $1 + 9 + 5$ and $2 + 9 + 4$
- Enter your triples of numbers in the rows, columns and diagonals of a 3 by 3 grid.
- Have you met anything like this before?
- Remember, in these puzzles and games you can't use 2 numbers to make 15, you can't use 4 numbers, it must be 3 numbers.

NEXT

- When you write down all the different combinations of 3 numbers that sum to 15:
 - How can you be sure that you have found all the combinations?
 - How many combinations of 3 numbers summing to 15 are there?
 - Have you met anything like this before?
 - Which numbers appear in 2 of these combinations?
 - Which numbers appear in 3 of these combinations?
 - Which numbers appear in 4 of these combinations?
- Can you think of any other games you know like this one?
- Make up a number game of your own.

1	2	3	4	5
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6	7	8	9
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Upper Primary

The learners should play the Make Ten and Fifteen Game if they have not played them before.

Now the learners should play Tic-Tac-Toe (Noughts and Crosses) and think about what is similar, and what is different, about Tic tac Toe and The Fifteen Game.

TIC TAC TOE – NOUGHTS AND CROSSES GAME for 2 players or 2 teams.



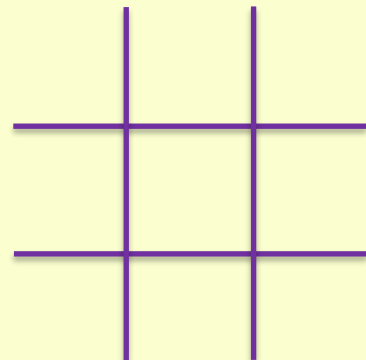
3 by 3 grid and coloured pens.

Take turns to write your symbol in one of the empty spaces.
One player writes X and the other writes O.

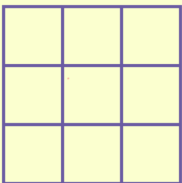
The winner is the first to get a line of 3 of his symbols.

Continue until someone wins or all spaces have been filled.

The game may end in a draw.



MAGIC SQUARE PUZZLE



Without telling the learners that there is a connection to the Fifteen Game, ask them to write the numbers 1 to 9 into a 3-row and 3-column array so that all the rows, columns and diagonals have the same total. But don't tell them what the total is.

Suggestions for Teaching

This **Magic Square Puzzle** is suitable for a mixed age group that includes primary age learners because it only involves simple addition and problem solving. Younger learners may sometimes succeed in this task before the older ones.

If you have a larger group then you could ask them to do this individually, then in pairs, then in fours, and to exchange and share ideas for methods and solutions.

You could give the hint that it helps if you find the total of $1 + 2 + 3 + \dots + 9$. Suggest that, to understand better how to find solutions, learners could write down all the sets of 3 digits from the 1, 2, 3, ...9 that add up to 15 without repeating any digit. The 8 solutions correspond to the 8 rows, columns and diagonals in a 3 by 3 array.

Learners may find what appear to be different solutions. Ask 'Are they different?' Discuss the fact that **there is only one solution** because all the solutions are rotations or reflections of each other.

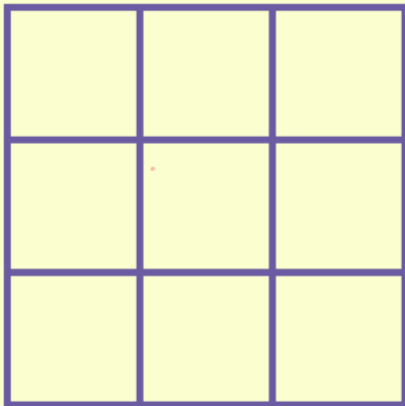
Learners who complete the Magic Square puzzle before the rest of the class should do the **UnMagic Square** puzzle where all the totals of rows, columns and diagonals are **different**. Here is an example. Can you find other solutions with different numbers in the centre?

3	4	5
6	9	2
1	8	7

MAGIC SQUARE GAME for 2 players or 2 teams



3 by 3 grid and cards numbered 1 to 9.



Stage 1: Take turns either to put one of the 9 cards into an empty square, or to move one of the cards that has already been placed to an adjacent empty square.

Stage 2: Take turns to swap 2 numbers.

The winner is the first player to make a line of 3 cards adding up to 15 (across, up and down or diagonally).

1	2	3	4	5
6	7	8	9	

Lower Secondary

Play the **Fifteen Game, Tic Tac Toe and Magic Squares Games, Then Play**

PAT THE PIG GAME a game for 2 players or 2 teams.



9 word-cards or simply pencil and paper.



Take it in turns to claim one of the 9 words.

The winner is the first to get all 3 occurrences of the same letter.

Aim to collect 3 words that contain the same letter and to stop your opponent from doing so. Continue until someone wins or all the words are taken. Words can only be used once.

GOT	AI
GAME	OIE
PET	PIG
PM	AT
MO	CARDS FOR PET PIG GAME

Suggestions for Teaching

Suggest to the learners that if they want to win this game, they might have a better chance if they use past experience. Suggest that, if they can recognise that the Pat the Pig Game is similar to any game they have played before, then they might use that knowledge to help them win this game. Ask them to work in pairs to answer the key questions and then arrange for each pair to challenge another pair to put into practice their ideas about the best way to win.

KEY QUESTIONS

- What is similar and what is different about the games Fifteen, Tic Tac Toe, Magic Square and Pat the Pig?
- What occur in sets of 9 in the games Fifteen, Tic Tac Toe, Magic Square and Pat the Pig?
- What occur in in sets of 8 the games?
- What is similar about winning the games?
- Have you found a winning strategy?
- Can you find a strategy so that you never lose this game?
- Can you explain your strategy?

GOT

AI

GAME

OIE

PET

PIG

PM

AT

MO

**CARDS FOR
PET PIG
GAME**

Upper Secondary

The challenge is to explain the equivalence of:

THE FIFTEEN GAME

3 BY 3 MAGIC SQUARES

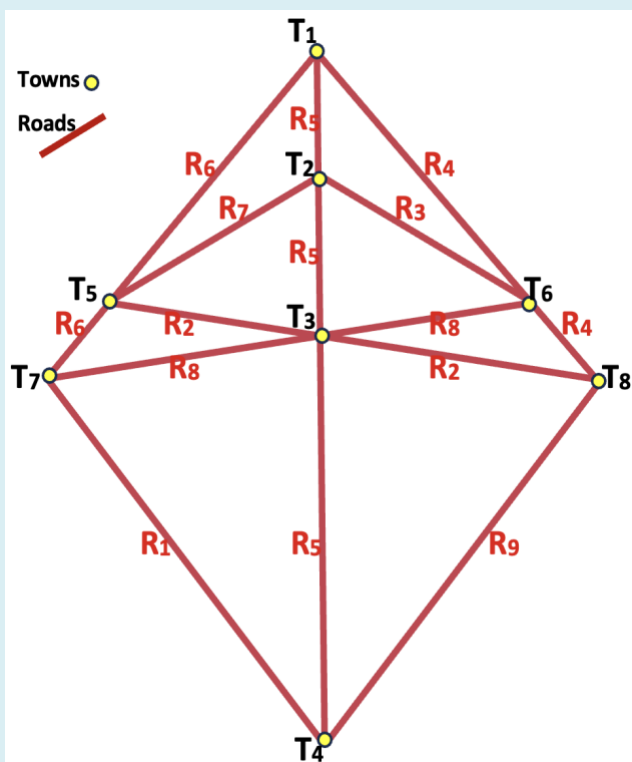
NOUGHTS AND CROSSES

PAT THE PIG GAME

SIEGE GAME

When you investigate the equivalence between one game and another, explain how the properties (features) in one game correspond to properties of the other game.

THE SIEGE GAME



If learners have not played the games before, they should play the **Fifteen Game**, **Tic Tac Toe**, **The Magic Square Game** and **Pat the Pig** which can be played using only pencil and paper. They should be encouraged to look for similarities between the games.

Then introduce the **Siege Game**. This is a game for two players. Each player takes a turn to claim a whole straight road (even though it may pass through several towns). To win you must be the first to block 3 roads going to, or through, the same town. You need the gameboard which is easy to copy so you can play this game using only pencil and paper. You can use different coloured pens to mark the roads you claim, or coloured counters. A larger version of the gameboard is shown on page xx.

Notice that there are 8 dots representing towns and each town lies on exactly 3 roads.

Notice that there are 9 straight lines representing roads going to or through the towns. Some roads connect 2 towns, some connect 3 towns and one connects four towns.

Suggestions for Teaching

It's important for learners to be able to gather all the information available when solving problems and to record that information in a clear and concise way. One of the best ways of summarising information and identifying connections between ideas, is to use a table. Learners could work in small groups to answer the key questions, and to compile a table to record their answers. Suggest that they draw up a table with a column for each game and use the rows to record all the information that they have about the different games: Tic tac Toe (Noughts and Crosses, the Fifteen Game, the Magic Square Game, Pat the Pig and Siege.

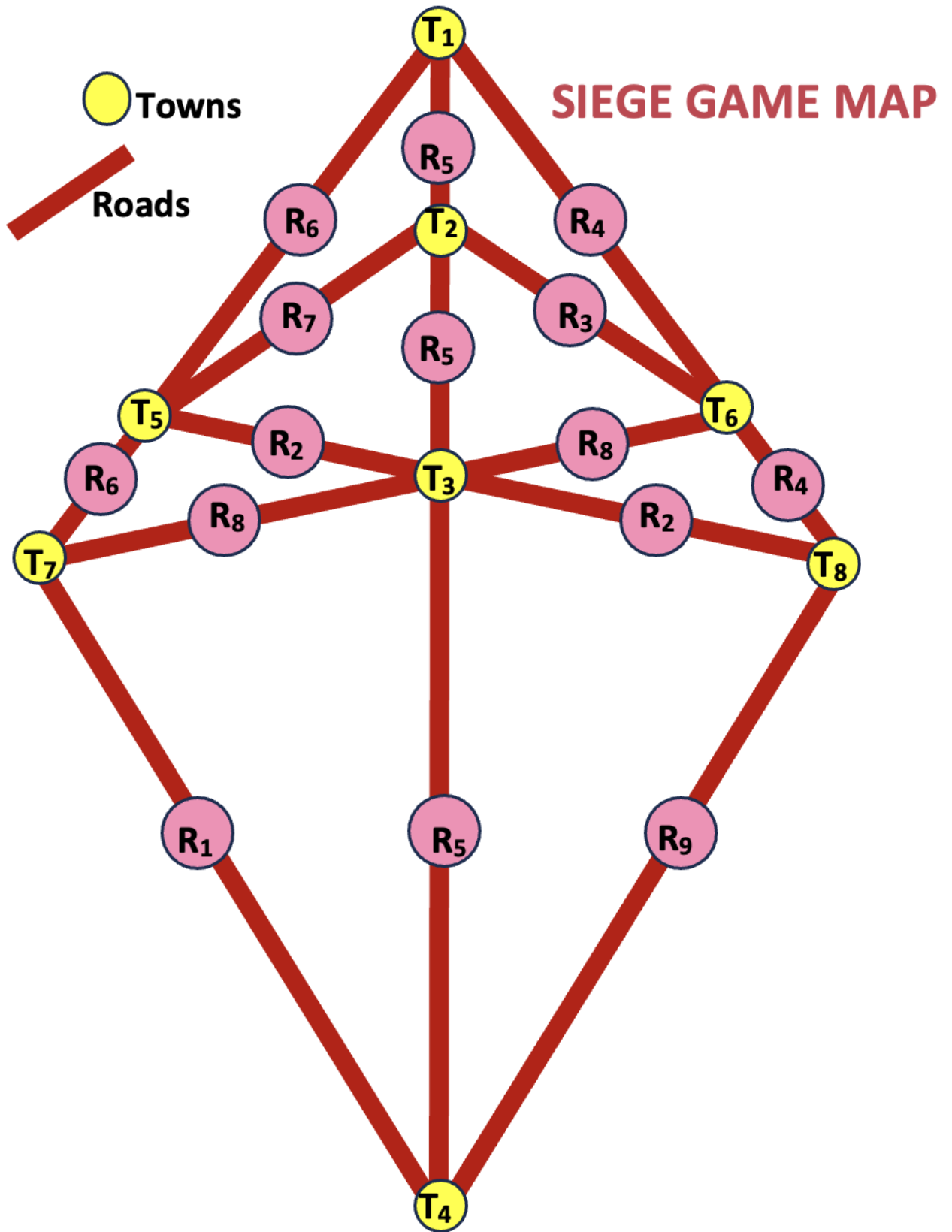
Key Questions for THE SIEGE GAME

- Is Siege like any other game that you have played before?
- Can you write down all combinations of three roads through the same town?

- Which roads appear in 2 of these combinations?
- Which roads appear in 3 of these combinations?
- Which roads appear in 4 of these combinations?
- What occur in a set of 9?
- What occur in a set of 8?
- Have you found a winning strategy?
- Is it possible to avoid losing this game?
- Is it an advantage to go first (or second?)
- Can you become an expert and never lose this game, although the game may sometimes end in a draw?

Then ask several groups to present their tables to the class. If flip chart or other large sheets of paper are available, then the groups could draw their tables on paper. Otherwise they should draw them on the board. Orchestrate a discussion and then ask the groups to improve their tables if they can do so.

Discuss the concept of isomorphism which means 'changing in the same way'. Splitting this Greek word, we interpret 'iso' to mean 'same' and 'morphism' to mean 'something changing'. Changes in one game match equivalent changes in the other games. Isomorphism carries with it the crucial information about the objects involved and the way that they are combined, while discarding irrelevant information. In this case the five contexts are to some extent irrelevant. The relationships and structure are the same in all five games. Isomorphism is very useful because, if you recognise the structure and have seen it in another form, you already know a lot about it so it saves work. This can be discovered by learners if they meet all these games.



Follow up

ULTIMATE TIC TAC TOE

Two players (X and O) play 9 simultaneous games of Tic Tac Toe on a 9 by 9 mega-board taking turns to put their mark on one of the 81 squares. X starts.

To win a player must win games on 3 mini-boards in a row.



A1	2	3	B			C		
4	5	6						
7	8	9						
D			E			F		
G			H			I		

The position by a player on a 3 by 3 mini-board forces the next player to play on the mini-board that is in the same relative position on the mega-board as shown in the example.

A	B	C
D	E	F
G	H	I

If the first move by X was at E8 in the mini-board E as shown, then O is forced to play

next on mini-board H shaded in red in the mega-board.

If O plays at H5 in the centre of

board H, then X is forced to play again somewhere on mini-board E.

When players win any one of the 9 mini-games, they claim that board and close it down by marking it with a large X or O. If a player has been forced to make the next move on a closed board, then they have a free choice for that move of any position on any other board.

Keep an eye on the big picture. Focus on certain mini-boards aiming to win on the larger board. In the example shown here, O has won with the vertical line of three Os on mini-boards C, F and I. Aim to divert your partner from playing on your chosen mini-boards.

Why introduce Ultimate Tic Tac Toe in lessons?

Playing this game develops logical thinking and the ability to see the 'big picture' in a situation, and to optimise actions by taking into account the interplay of many different factors that could influence the outcome. Learners need to develop these skills if they are to thrive in adult life and once introduced to this game some learners will want to go on playing it out of school. The incentive to win leads to concentration and to thinking several moves ahead to plan a strategy, asking themselves questions like: "If my opponent made that move, how would it affect my play?" ... "What if...?", Then what...?"

Learning objectives

In doing this activity students will have an opportunity to:

- think mathematically;

- think, plan and act strategically.

Generic competences

- see the 'big picture' in a situation where many interacting factors influence the outcome;
- visualize and interpret images to represent concepts and situations;
- work systematically to investigate all possible cases;
- develop the life skill of winning and losing with consideration for others.

Key Questions for Ultimate Tic Tac Toe

- Can you decide at the outset which line of 3 mini-boards you are aiming for and play so that your opponent is diverted away from your chosen boards?
- Would forcing your opponent to play on a certain mini-board give you a better chance of winning on another mini-board and how can you use this to your advantage?
- Would it be to your advantage to sacrifice a mini-board to your opponent so that you can win a more important mini-board yourself?

STRATEGIC THINKING AND GAME THEORY

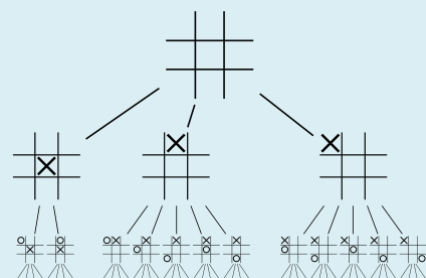
Mathematical Game theory has been described as the science of decision making. It uses mathematics that is more advanced than school mathematics and it has been recognised as an important tool in many fields including economics, business, the sciences, computer science and the social sciences. Game theory is especially important now as computer algorithms involving big data analysis and artificial intelligence play a big part in decision making in many aspects of modern life.

There is competitive and cooperative game theory. Optimal strategies are analysed assuming that groups of individuals can agree to act according to agreed rules and can successfully enforce the agreement.

There is classical game theory and combinatorial game theory. Classical game theory involves games of chance in which players move or place bets simultaneously as in roulette, poker or rock, paper scissors. By contrast, in combinatorial game theory, players have full knowledge of all the possible moves and all aspects of the game as in games like chess and ultimate tic tac toe. In these games players take turns to make moves and chance is not usually a factor.

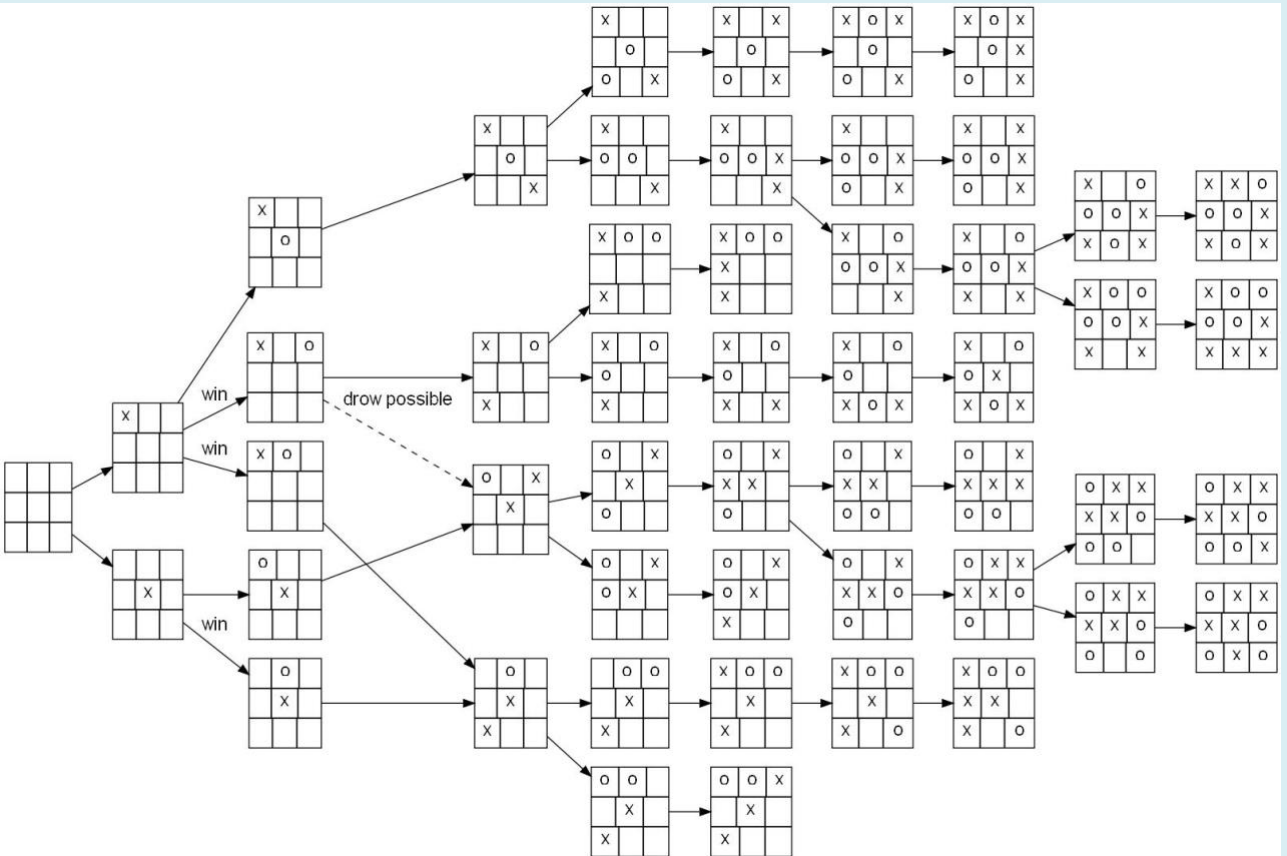
Visualizing the game tree:

- A game tree shows all possible sequences of moves. For example, the diagram on the right shows the first two plays in a simple Tic Tac Toe game.



b. The middle of an edge is not a good starting position so the tree diagram below, from Wikipedia Commons, shows a game analysis of the basic Tic Tac Toe game starting in the other two positions. It shows that, with optimal play, the game will always end in a draw.

c. This gives us an insight into how imagining a game tree, or predicting future moves, for



Ultimate Tic Tac Toe, where each move determines the next mini-board, is considerably more complicated.

SOLUTIONS

Combinations that add up to 15: $1 + 5 + 9$ $1 + 6 + 8$
 $2 + 4 + 9$ $2 + 5 + 8$ $2 + 6 + 7$
 $3 + 4 + 8$ $3 + 5 + 7$
 $4 + 5 + 6$

Note that 5 appears in 4 of the winning triples, more often than any other number, so the player going first should choose 5 because it gives an advantage.

With skilful play by both players, nobody should lose this game. It is important to let people discover this for themselves. It may take a long time and playing lots of games.

Number	1	2	3	4	5	6	7	8	9
In how many triples?	2	3	2	3	4	3	2	3	2

8 R ₈	1 R ₁	6 R ₆	PET	PM	PIG
3 R ₃	5 R ₅	7 R ₇	AT	GAME	AI
4 R ₄	9 R ₉	2 R ₂	GOT	MO	OIE

In the **3 by 3 Magic Square** all rows, columns and diagonals add up to 15. The number 5 occurs in 4 combinations so it must go at the centre. The even numbers 2, 4, 6 and 8, go at the corners because they occur in 3 combinations. The odd numbers 1, 3, 7 and 9 go at the mid-edges as they occur in 2 combinations.

The roads R₁, ... R₉ in the Siege Game are numbered to show the correspondence of 3 numbers adding to 15 and 3 roads blocking a town.

The connection to the 3 by 3 array also links the Fifteen Game to Tic Tac Toe or Noughts and Crosses the Pet Pig Game and the Siege Game.

Tables are important for organising and recording information. Putting information into a table often makes patterns clear, shows up gaps in the information, or shows some important feature of the problem being worked on. Sometimes learners can't solve a problem because they can't sort out the necessary information and a table helps them to sort it out.

TABLE OF CORRESPONDENCES BETWEEN THE GAMES

	GAMES				
	FIFTEEN GAME	MAGIC SQUARE GAME	PAT THE PIG GAME	SIEGE GAME	TIC TAC TOE (NOUGHTS AND CROSSES)
OBJECTS	9 cards numbered 1 to 9	Numbers 1 to 9	9 words	9 roads	9 positions on 3 by 3 grid
	8 sets of 3 numbers with a total of 15	8 lines in the grid horizontally, vertically and diagonally.	8 sets of 3 words with a letter in common	8 towns, each at a junction of 3 roads.	8 lines in the grid horizontally, vertically and diagonally.
OPERATIONS LEADING TO WINNING THE GAME	A player collecting a set of 3 numbers with a total of 15	A player completing a line in the grid with a total of 15	A player collecting a set of 3 words with a letter in common	A player claiming all 3 roads through one of the 8 towns	A player completing a line in the grid with 3 of his symbols X or O

Follow up

4 by 4 and 5 by 5 Magic Squares: <https://aiminghigh.aimssec.ac.za/picture-puzzler/>

Symmetries in Magic Squares:

<https://aiminghigh.aimssec.ac.za/durers-magic-square/>



Go to the **AIMSSEC AIMING HIGH** website for lesson ideas, solutions and curriculum links: <http://aiminghigh.aimssec.ac.za>

Subscribe to the **MATHS TOYS YouTube Channel**

<https://www.youtube.com/c/mathstoys>

Download the whole AIMSSEC collection of resources to use offline with the AIMSSEC App see <https://aimssec.app> Find the App on Google Play.

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and school years up to Secondary 5 in East Africa.

New material will be added for Secondary 6.

For resources for teaching A level mathematics (Years 12 and 13) see <https://nrich.maths.org/12339>

Mathematics taught in Year 13 (UK) & Secondary 6 (East Africa) is beyond the SA CAPS curriculum for Grade 12

	Lower Primary Approx. Age 5 to 8	Upper Primary Age 8 to 11	Lower Secondary Age 11 to 15	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13