AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

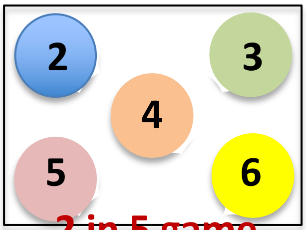
The ODDS Game Inclusion and Home Learning Guide provides related activities for lessons in school and home learning for all ages and learning stages from pre-school to school-leaving, on the Common Theme PROBABILITY SAMPLE SPACES

Choose what seems suitable for the age or attainment level of your learners.

Watch the videos: GTEN workshop https://youtu.be/aOoIdvJMUi8

AIMING HIGH LEARNING PACK https://aiminghigh.aimssec.ac.za/odds-and-evens/

ODDS GAME

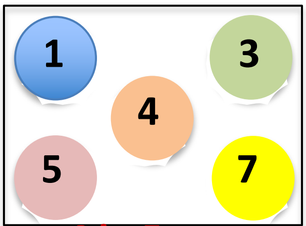


2 in 5 game

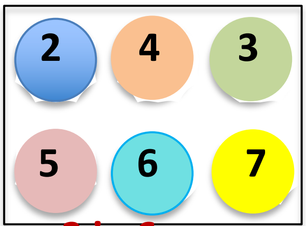
In the 2 Odds in 5 game you pick out two balls at random from a bag containing 5 balls numbered 2, 3, 4, 5 and 6.

If the total is odd you win. If it is even you lose.
Are you equally likely to win or to lose (a fair game)?
In the 3 games shown below, the rules are the same.
Which set of balls would you choose to give yourself the best chance of winning?

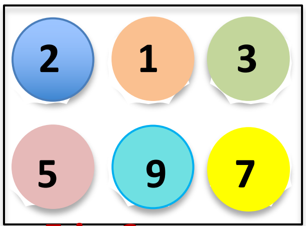
Do your own experiments. Make number cards and put them in an envelope or bag.
For interactive computer simulations of the games see http://nrich.maths.org/4308



4 in 5 game



3 in 6 game



5 in 6 game

HELP

2 in 5 Game		FIRST CHOICE				
		2	4	6	3	5
SECOND CHOICE	2	X			5	
	4	6	X			
	6		10	X		
	3				X	8
	5					X

Play the 2 odds in 5 game several times and record the results. You are conducting trials in a probability experiment.

Fill in a table colouring the odd and even totals in contrasting colours. Put crosses to show you can't choose the same ball twice.

This is called **a sample space diagram**.

The sample space diagram shows that, when you choose 2 cards at random there are exactly 20 possible outcomes, that is 20 elements in the sample space.

NEXT

None of the sets looked at so far gives a fair game.
Can you find a set of numbers that would give a fair game?

PROOF SORTING EXERCISE FOR YEAR 12 AND 13 STUDENTS.

RED/BLUE and ODDS/EVENS GAMES Imagine we play the game with different objects (counters, cards, balls) of two types (numbered objects or objects of two colours red and blue). **(A)**

This is the 1st statement of the proof, and the other 9 statements are jumbled up. Cut out the strips and re-construct the proof by arranging the remaining statements in the correct order.

We have proved that: the game will be a fair game if and only if $(a-b)^2 = a + b$ **(B)**

- the total number of objects (i.e. $a + b$) has to be a square number, say n^2;
- the difference between the number objects of the two types (i.e. $a - b$) is n.

That is:

$$a + b = n^2 \quad (1)$$

$$\text{and} \quad a - b = n \quad (2)$$

We have proved that a necessary and sufficient condition for a game of this sort to be a fair game is that a and b must be consecutive triangle numbers. Consequently the total number of objects ($a + b$) is a square number n^2. **(C)**

This suggests that a and b are consecutive triangle numbers as the rule for generating the sequence of triangle numbers is that the difference between the nth and (n-1)th triangle number is equal to n, and their sum is the square number n^2. (e.g. $T_7 - T_6 = 7$ and $T_7 + T_6 = 7^2$) **(D)**

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	a	b
a	$a^2 - a$	ab
b	ab	$b^2 - b$

	R_1	R_2	R_3	B_1	B_2	B_3	B_4	B_5	B_6
R_1	x	S	S	D	D	D	D	D	D
R_2	S	x	S	D	D	D	D	D	D
R_3	S	S	x	D	D	D	D	D	D
B_1	D	D	D	x	S	S	S	S	S
B_2	D	D	D	S	x	S	S	S	S
B_3	D	D	D	S	S	x	S	S	S
B_4	D	D	D	S	S	S	x	S	S
B_5	D	D	D	S	S	S	S	x	S
B_6	D	D	D	S	S	S	S	S	x

To Prove This is a sample space diagram (also called a contingency table or 2-way table). **(E)**

It shows the total numbers of possible outcomes, grey for one outcome (e.g. different or odd) yellow for the other outcome (e.g. same or even).

Notation: Label the number of objects of the different types as a and b with $a \geq b$. **(F)**

The symbol $\Leftrightarrow$ means that the logical argument works both ways, the statement and its converse are both true, it's an 'if and only if' argument.

Adding the equations (1) and (2) : $2a = n^2 + n \iff a = \frac{1}{2}(n^2 + n) \iff a = \frac{1}{2}n\,(n+1)$ **(G)**

This formula gives a as the n^{th} triangular number.

Similarly, b is the $(n-1)^{th}$ triangular number $\frac{1}{2}n\,(n-1)$.

If a and b are any two consecutive triangle numbers, the reverse argument holds.

For a fair game we must have the probability of one outcome equal to the probability of the other outcome; that is: $(a^2 - a) + (b^2 - b) = ab + ab$ *(from the sample space diagram)* **(H)**

$$\iff a^2 - 2ab + b^2 = a + b \qquad \textit{(by rearranging this expression)}$$

$$\iff (a-b)^2 = a + b$$

Conclusion: Isomorphic games. The proof applies to all games that involve randomly picking two objects from a bag of objects of two different types where winning or losing corresponds to picking objects of the same or different types, for example, with red corresponding to odd and blue corresponding to even. The games are fair if and only if the number of objects of each type are consecutive triangle numbers a and b (so that $a + b = n^2$ and $a - b = n$). **(I)**

Conjecture: The game will be a fair game if and only if the numbers of objects of each type are consecutive triangle numbers e.g. $a = 6$ and $b = 3$. In this case the total number of objects is a square number. **(J)**

INCLUSION AND HOME LEARNING GUIDE

THEME: Probability Sample Spaces – Counting all possible outcomes

For Lower Secondary the Diagnostic Quiz is used at the end of a lesson, or series of lessons, on sample spaces to assess what has been learned.

For Upper Secondary the Diagnostic Quiz is used at the start of a lesson, or series of lessons, on sample spaces formatively to assess prior knowledge.

Diagnostic Quiz

This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 for C and 4 for D".
2. Notice how the learners respond. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. It is important for learners to explain the reason for their answer to clarify their own thinking and to practise communication skills.
4. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
5. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.**
6. The concept is needed for the lesson to follow, so explain the right answer or give a remedial task.

Two dice are rolled together. The two numbers are added together.

Find the probability of getting a total of less than 4 or more than 10.

Die 1

Die 2		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

A	B	C	D
$\frac{1}{6}$	$\frac{12}{48}$	$\frac{1}{3}$	$\frac{1}{12}$

The correct answer is **A.** There are 6 possible outcomes out of 36.

Common Misconceptions: B. May be a guess

C. These learners might have included scores of 4 & 10 or misunderstood 'less than'.

D. May be a guess https://diagnosticquestions.com

Lower Secondary for Years 7 – 10

Resources: You need small objects of two different colours that are the same in all respects other than colour, such as counters or buttons or the cards cut out from page 5, and a bag or envelope.

Introduce both the Red Blue Games and the Odds Games. The learners should play the games and try to decide whether the games are fair or whether they have a strategic advantage when they choose to be on one team rather than the other. The learners should then fill out probability space diagrams and compare the different games. They should discuss what is the same and what is different between the Odds Games and the Colours games and notice that each game is like the other game in disguise. The teacher should introduce them to the idea of ISOMORPHISM between two mathematical systems.

COLOURS GAME WITH 5 RED CARDS AND 5 BLUE CARDS IN THE BAG.

Play this game and investigate what happens. Record the results for each draw as 'same colour' or 'different colour'.

Many learners will discover that this is NOT a fair game. They are more likely to get 2 cards of different colours than to get 2 cards of the same colour. The teacher should not give away this 'secret winning strategy' so that more and more members of the class will realize **for themselves** that they should choose to be 'different'. The teacher should encourage them to want to find out why.

To understand why players are more likely to draw two cards of different colours than two of the same colour, learners need to understand how to fill in a 2-way table showing all possible outcomes.

Ask the learners to write S for 'same' and D for 'different' in the empty squares in the diagram.

Ask the learners What do the red crosses on the diagonal line tell you?

	R_1	R_2	R_3	R_4	R_5	B_1	B_2	B_3	B_4	B_5
R_1	x									
R_2		x								
R_3			x							
R_4				x						
R_5					x					
B_1						x				
B_2							x			
B_3								x		
B_4									x	
B_5										x

Year 7 learners may not have met 2-way table like the one shown. Introduce the table and how to fill it in. Ask the learners to study the table and try to find out why it is more likely to draw two cards of different colours than two of the same colour.

Then the learners should play more games varying the proportions of red and blue cards and deciding whether or not they are fair games. Ask them to make a list of the total numbers of objects for fair games.

RED BLUE GAME INVESTIGATION

What do you notice about the small squares (cells) covered by the yellow triangles?
What do the covered cells represent?
How many are there?

	R_1	R_2	R_3	R_4	R_5	B_1	B_2	B_3	B_4	B_5
R_1	x									
R_2		x								
R_3			x							
R_4				x						
R_5					x					
B_1						x				
B_2							x			
B_3								x		
B_4									x	
B_5										x

What do you notice about the small squares (cells) covered by the grey squares?
What do the covered cells represent?
How many are there?

	R_1	R_2	R_3	R_4	R_5	B_1	B_2	B_3	B_4	B_5
R_1	x									
R_2		x								
R_3			x							
R_4				x						
R_5					x					
B_1						x				
B_2							x			
B_3								x		
B_4									x	
B_5										x

RED BLUE GAME SAMPLE SPACE DIAGRAM SUMMARY

The SAMPLE SPACE is the set of all possible outcomes. In this case it is the set the 90 possible results when you draw 2 cards from a bag with 5 red and 5 blue cards.

	R_1	R_2	R_3	R_4	R_5	B_1	B_2	B_3	B_4	B_5
R_1	X	S	S	S	S	D	D	D	D	D
R_2	S	X	S	S	S	D	D	D	D	D
R_3	S	S	X	S	S	D	D	D	D	D
R_4	S	S	S	X	S	D	D	D	D	D
R_5	S	S	S	S	X	D	D	D	D	D
B_1	D	D	D	D	D	X	S	S	S	S
B_2	D	D	D	D	D	S	X	S	S	S
B_3	D	D	D	D	D	S	S	X	S	S
B_4	D	D	D	D	D	S	S	S	X	S
B_5	D	D	D	D	D	S	S	S	S	X

Each cell shows the result for picking 2 cards at random, first one, then another, from a bag containing 5 red cards and 5 blue cards.

The crosses show you can't pick the same card twice.

There are 90 possibilities: 40 with the same-coloured cards, 50 with different coloured cards. Picking different coloured cards is more likely than the same colour.

The probability of 'same' is $\frac{40}{90} = \frac{4}{9}$.

The probability of 'different' is $\frac{50}{90} = \frac{5}{9}$

It is NOT A FAIR GAME.

Introduce the Odds Games as on page 1 and in the worksheet. The 4 games are all played with the same rules. Are any of the games fair?

Find a set of numbers that gives a fair game. How do you know which sets of numbers give fair games?

Explore some of the properties of odd and even numbers. Tell the class to close their eyes and keep closed until you tell them to open their eyes. Tell them:
ODD + ODD
Think of any two odd numbers. Add them. Is the sum odd or even?
Think of another two odd numbers. Add them. Is the sum odd or even?
Do this a few more times. Ask: What do you notice?

EVEN + EVEN
Think of any two even numbers. Add them. Is the sum odd or even?
Think of another two even numbers. Add them. Is the sum odd or even?
Do this a few more times. Ask: What do you notice?
ODD + EVEN
Think of any two numbers, one odd one even. Add them. Is the sum odd or even?
Think of another two numbers, one odd one even. Add them. Is the sum odd or even?
Do this a few more times. Ask: What do you notice?

WHAT IS THE SAME AND WHAT IS DIFFERENT ABOUT THESE TWO DIAGRAMS?

RED BLUE GAME SAMPLE SPACE 2-WAY TABLE

	R_1	R_2	R_3	B_1	B_2	B_3	B_4	B_5	B_6
R_1	X	S	S	D	D	D	D	D	D
R_2	S	X	S	D	D	D	D	D	D
R_3	S	S	X	D	D	D	D	D	D
B_1	D	D	D	X	S	S	S	S	S
B_2	D	D	D	S	X	S	S	S	S
B_3	D	D	D	S	S	X	S	S	S
B_4	D	D	D	S	S	S	X	S	S
B_5	D	D	D	S	S	S	S	X	S
B_6	D	D	D	S	S	S	S	S	X

ODD EVEN GAME SAMPLE SPACE 2-WAY TABLE

	O_1	O_2	O_3	E_1	E_2	E_3	E_4	E_5	E_6
O_1	X	E	E	O	O	O	O	O	O
O_2	E	X	E	O	O	O	O	O	O
O_3	E	E	X	O	O	O	O	O	O
E_1	O	O	O	X	E	E	E	E	E
E_2	O	O	O	E	X	E	E	E	E
E_3	O	O	O	E	E	X	E	E	E
E_4	O	O	O	E	E	E	X	E	E
E_5	O	O	O	E	E	E	E	X	E
E_6	O	O	O	E	E	E	E	E	X

For all odd and all even numbers: even + even = even
odd + odd = even
and odd + even = odd

Upper Secondary

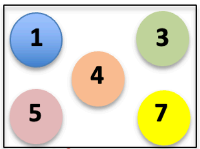
Many ‘real world’ situations (such as clinical trials in medical research) cannot be analysed theoretically and must depend on experimental probability, with a large number of trials on subjects drawn randomly from the population, so as to provide a mathematical model for the real situation.

In doing this activity you will explore two types of probability: experimental and theoretical by playing the games, trying to discover by experimenting whether they are fair games, and then analysing the games to work out the theoretical probabilities of drawing numbers adding to odd and even totals. You will meet isomorphism (mathematical systems that look different but are really the same) and you will apply some simple algebra to prove a surprising result.

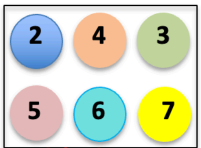
Work through the activities on page 1, if possible with a partner or group. Use numbered cards in a bag or use the computer simulation

http://nrich.maths.org/4308. The computer simulation generates lots of experimental data quickly, freeing time to focus on predictions, analysis and justifications. To calculate the theoretical probabilities use sample space diagrams like the one in the HELP box.

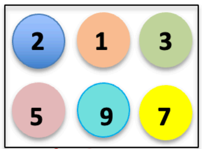
2 in 5 game



4 in 5 game

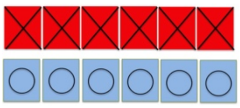


3 in 6 game

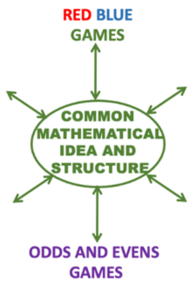


5 in 6 game

Prove to yourself that none of these games are fair games because the probability of getting an odd score is not equal to the probability of getting an even score.



Play the Red and Blue Games (see page 3. Explain how these games are isomorphic to the Odds Games. ‘Isomorphic’ means that the two systems have exactly the same structure. The only difference is the context of 2 colours mimicking the odd and even numbers. The word is derived from the Greek ‘iso’ meaning the same and ‘morph’ meaning change.



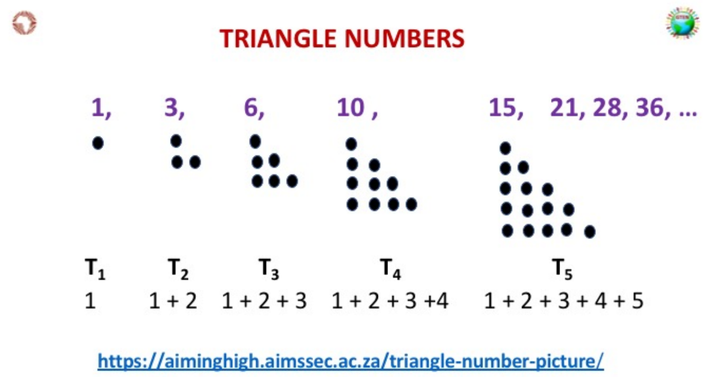
Key questions

- How can you decide if a game is fair?
- What are the most efficient methods for recording possible combinations?
- How can we make this difficult task (of finding a fair game) more manageable?
- Make a list of the total number of objects used when these are fair games. What properties do these numbers have?

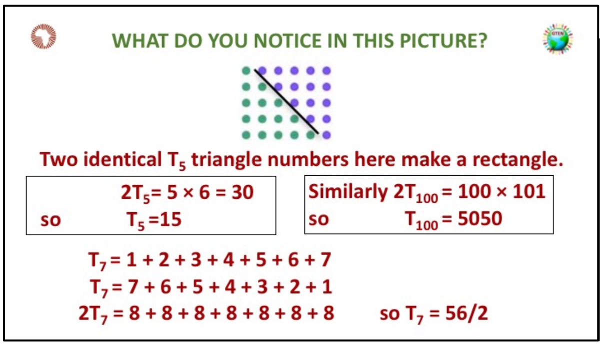
Formulate conjectures about what conditions lead to a fair game and try to prove your conjectures.

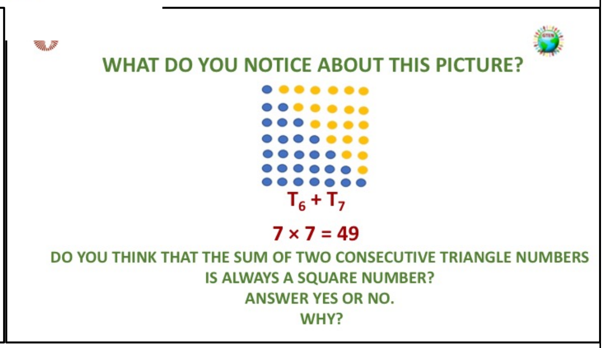
Hint 1: List the total number of objects put in the bag for all the fair games. What do you notice about these totals?

Hint 2: Investigate the very simplest arithmetic series:



Look closely, what do you notice about the diagrams of triangle numbers in the two illustrations below?





Numbers	1	2	3	4	5	…	99	100
Reversed	100	99	98	97	96	…	2	1
Totals	**101**	**101**	**101**	**101**	**101**		**101**	**101**

What do you notice about this table and the picture above it on the left?

Does it tell you how to prove the formula for the sum of n terms of the arithmetic series 1 + 2 + 3 + 4 +… ?

Play the Red and Blue Games (see page7). Explain how these games are isomorphic to the Odds Games. ‘Isomorphic’ means that the two systems have exactly the same structure. The only difference is the context of 2 colours mimicking the odd and even numbers. The word is derived from the Greek ‘iso’ meaning the same and ‘morph’ meaning change.

Now is the time for abstract thinking and relying on the power of algebra

Proof Sorting Exercise for Year 12 and 13 Students
Cut out the strips on page 5. Arrange them in the correct

RED/BLUE and ODDS GAMES Imagine we play the game with different objects (counters, cards, balls) of two types (two colours or numbered even and odd.) Can you prove the conjecture: The game will be a fair game if and only if the numbers of objects of each type are consecutive triangle numbers and the total number of objects is a square number.

Cut out the strips on page 4 and re-construct the proof of the conditions for the numbers of odd and even cards to lead to a fair game by arranging the statements in the correct order.

Why do this activity?

In this activity learners can explore and discuss two types of probability: experimental and theoretical. They can experience teamwork in trying to find an example of a fair game of this type and trying to formulate conjectures about what conditions might lead to a fair game. Older learners can work on the challenge of proving the conjectures.

Learning objectives

In doing this activity students will have an opportunity to:

- conduct probability experiments and record results;
- experience and discuss random sampling;
- learn to use a sample space diagram to show all the elements in a sample space and to analyse the theoretical probabilities of different events;
- reflect on the differences between experimental probability and theoretical probability;
- find and prove conjectures about conditions for a fair game.

Generic competences

In doing this activity students will have an opportunity to:

- conduct probability experiments, reflect on the differences between experimental probability and theoretical probability and discuss the types of applications that require the use of one or the other;
- work as a team towards a common goal;
- **communicate** in writing, speaking and listening:
    - exchange ideas, criticize, and present information and ideas to others;
    - analyze, reason and record ideas effectively.

SOLUTION

2 in 5 game

Probability of an even sum is $\frac{8}{20} = \frac{2}{5}$.
Probability of an odd sum is $\frac{12}{20} = \frac{3}{5}$.
So it is not a fair game.

	2	4	6	3	5
2	X	6	8	5	7
4	6	X	10	7	9
6	8	10	X	9	11
3	5	7	9	X	8
5	7	9	11	8	X

4 in 5 game

Probability of an even sum is $\frac{12}{20} = \frac{3}{5}$.
Probability of an odd sum is $\frac{8}{20} = \frac{2}{5}$.
So it is not a fair game.

	1	3	5	7	6
1	X	4	6	8	7
3	4	X	8	10	9
5	6	8	X	12	11
7	8	10	12	X	13
6	7	9	11	13	X

2 in 6 game

Probability of an even sum is $\frac{14}{30} = \frac{7}{15}$.
Probability of an odd sum is $\frac{16}{30} = \frac{8}{15}$.
So it is not a fair game.

	2	4	6	8	3	5
2	X	6	8	10	5	7
4	6	X	10	12	7	9
6	8	10	X	14	9	11
8	10	12	14	X	11	13
3	5	7	9	11	X	8
5	7	9	11	13	8	X

5 in 6 game

Probability of an even sum is $\frac{21}{30} = \frac{7}{10}$.
Probability of an odd sum is $\frac{9}{30} = \frac{3}{10}$.
So it is not a fair game.

	1	3	5	7	9	4
1	X	4	6	8	10	5
3	4	X	8	10	12	7
5	6	8	X	12	14	9
7	8	10	12	X	16	11
9	10	12	13	16	X	13
4	5	7	9	11	13	X

Odds Games Theorem and Proof

RED/BLUE and ODDS GAMES Imagine we play the game with different objects (counters, cards, balls) of two types (two colours red and blue or numbered even and odd).

Notation: Label the number of objects of the different types as a and b with $a \geq b$.

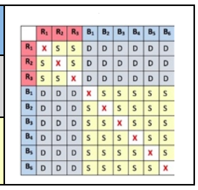
The symbol ⇔ means that the logical argument works both ways, the statement and its converse are both true, it's an 'if and only if' argument.

Conjecture: The game will be a fair game if and only if the numbers of objects of each type are consecutive triangle numbers. In this case the total number of objects is a square number, e.g. $a = 6$ and $b = 3$.

To Prove:

This is the sample space diagram (2-way table). It shows the total numbers of possible outcomes, with grey for one outcome (e.g. different or odd) and yellow for the other (e.g. same or even).

	a	b
a	$a^2 - a$	ab
b	ab	$b^2 - b$



For a fair game we must have the probability of one outcome equal to the probability of the other outcome; that is: $(a^2 - a) + (b^2 - b) = ab + ab$ *(from the sample space diagram)*

$\Leftrightarrow \quad a^2 - 2ab + b^2 = a + b$ *(by rearranging this expression)*

$\Leftrightarrow \quad (a - b)^2 = a + b$

We have proved that: the game will be a fair game if and only if $(a - b)^2 = a + b$

- the total number of objects (i.e. $a + b$) has to be a square number, say n^2;
- the difference between the number objects of the two types (i.e. $a - b$) is n

  that is $a + b = n^2$ (1)

  and $a - b = n$ (2)

This suggests that a and b are consecutive triangle numbers because the rule for generating the sequence of triangle numbers is that the difference between the n^{th} and $(n - 1)^{th}$ triangle number is equal to n, and their sum is a square number.

Adding the two equations: $2a = n^2 + n \Leftrightarrow a = ½ (n^2 + n) \Leftrightarrow a = ½n (n+1)$

This formula gives a as the n^{th} triangular number.

Similarly, b is the $(n - 1)^{th}$ triangular number $½n (n - 1)$.

If a and b are any two consecutive triangle numbers, the reverse argument holds.

We have proved that a necessary and sufficient condition for a game of this sort to be a fair game is that a and b must be consecutive triangle numbers, in which case the total number of objects $(a + b)$ is a square number n^2.

Conclusion: Isomorphic Games. The proof applies to all games that involve randomly picking two objects from a bag of objects of two different types where winning or losing corresponds to picking objects of the same or different types. The games are fair if and only if the number of objects of each type are consecutive triangle numbers a and b ($a + b = n^2$ and $a - b = n$). The conclusion is counter-intuitive because if $a = b$ the game is never fair.

Follow up

The activity In a Box https://aiminghigh.aimssec.ac.za/in-a-box/ offers another context for exploring exactly the same game and underlying mathematical structure. Use it as a follow-up a few weeks after working on Odds & Evens.

Also see Special Sums https://aiminghigh.aimssec.ac.za/special-sums/

Red or Black: https://aiminghigh.aimssec.ac.za/red-or-black-game/

Nines or Tens https://aiminghigh.aimssec.ac.za/nines-and-tens/

Two Aces https://aiminghigh.aimssec.ac.za/two-aces/

In the Bag https://aiminghigh.aimssec.ac.za/in-the-bag/

Twos Company https://aiminghigh.aimssec.ac.za/twos-company/

Same Sweets https://aiminghigh.aimssec.ac.za/same-sweets