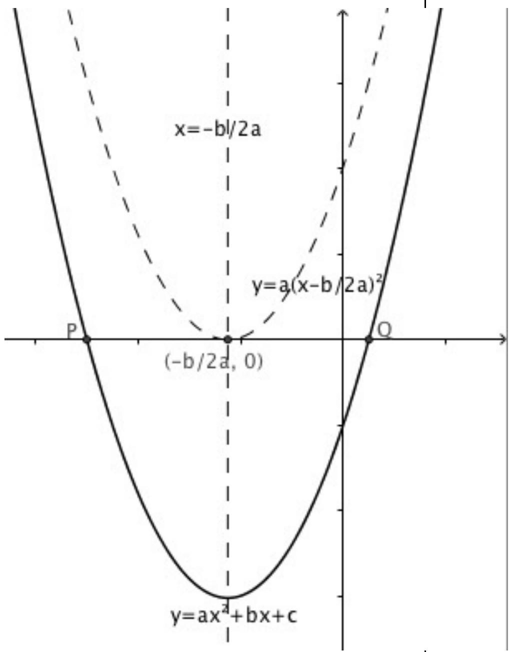
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

Solving Quadratic Equations and Transforming Parabolas

Thinking at the same time about the algebra and about transformations of the graph of the quadratic function $y = x^2$, we see how the formula for the solutions of a quadratic equation $ax^2 + bx + c = 0$ arise from the coordinates of the point where the graph of $y = ax^2 + bx + c$ cuts the x-axis. Replace the red question marks below.

To solve the quadratic equation $ax^2 + bx + c = 0$ we rearrange it to complete the square.	Think about where the graph of $y = ax^2 + bx + c$ cuts the y axis.	
$a(x+?)^2 + c - \frac{b^2}{4a} = 0$	Rearranging the equation.	
$a(x+?)^2 - \frac{b^2-?}{4a} = 0$	The graph is a translation of $y=x^2$ by $-b/2a$ in the x direction and $\frac{b^2-4ac}{4a}$ in the negative y direction so it is symmetrical about the line x= ?	
$(x+?)^2 = \frac{b^2-?}{4a^2}$		
Next take the square roots of both sides $(x+?) = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$	The roots of the equation are given by the x coordinates of the points P and Q which are symmetric on either side of the line x = ?	
The roots of the equation are $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$	P = ? Q = ?	

Example

To solve $2x^2 + 9x - 3 = 0$ by completing the square the equation can be rearranged to give

$$2\left[(x+?)^2 - \frac{3}{2} - \frac{81}{16}\right] = 0$$

$$2[(x+?)^2 - ?] = 0$$

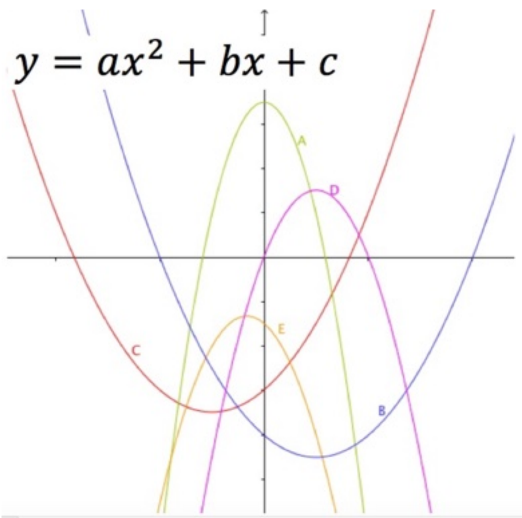
$$(x+?)^2 = ?$$

So the solutions are:

$x = -\frac{9}{4} \pm \frac{\sqrt{105}}{4} = \frac{-9 \pm 10.247}{4}$ = ? and ? (to 3 decimal places)

PROOF SORTER: Give a proof of the formula for the solutions of a quadratic equation by putting these statements in order:

$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$
Divide both sides by a: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
Rewrite the right-hand side with common denominator $4a^2$
The solutions are $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$
Add $(\frac{b}{2a})^2$ to both sides: $(x + \frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$
Subtract $\frac{c}{a}$ from both sides: $x^2 + \frac{b}{a}x = -\frac{c}{a}$
Subtract $\frac{b}{2a}$ from both sides
$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$
Complete the square: $(x + \frac{b}{2a})^2 - (\frac{b}{2a})^2 = -\frac{c}{a}$



HELP How do the two points where the graph cuts the x-axis relate to the axis of symmetry of the graph and the solutions of the quadratic equation $ax^2 + bx + c = 0$?

Match the graphs in the diagram to the following descriptions and give reasons for your decisions.

1. $y = ax^2 + bx + c$ if $a > 0, b > 0$ and $c < 0$
2. $y = ax^2 + bx + c$ if $a < 0, b = 0$ and $c > 0$
3. $y = ax^2 + bx + c$ if $a < 0, b < 0$, and $b^2 - 4ac < 0$
4. $y = a(x + p)^2 + q$ if $p < 0, q < 0$ and the x-intercepts have different signs.
5. $y = a(x + p)^2 + q$ if $a < 0, p < 0, q > 0$ and one root is zero.

https://aiminghigh.aimssec.ac.za/quadratic-functions/

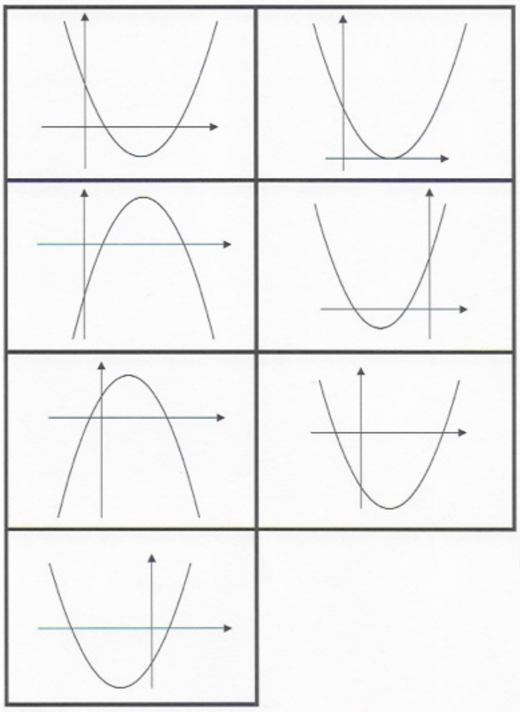
NEXT

A. **What if $b^2 - 4ac < 0$?**

What if the graph of the function does not intersect the x-axis?

See the video 'What are Numbers' https://youtu.be/s7uffFGhFLM This takes you beyond school mathematics and it explains that, all quadratic equations have solutions because, in addition to one dimensional real numbers (represented by points on the real number line) there are also two dimensional complex numbers (represented by points in the plane).

B. Match these equations to their graphs.



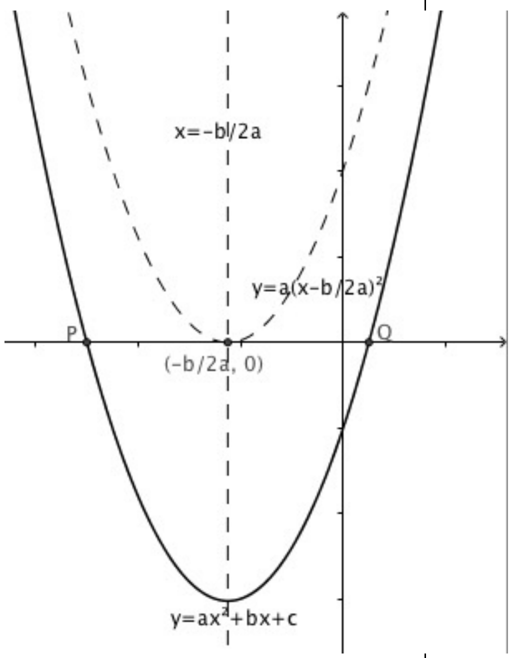
$y = x^2 + 6x - 16$	$y = x^2 - 8x + 16$
$y = 8 - x^2 + 2x$	$y = 6x - x^2 - 8$
$y = x^2 - 10x + 16$	$y = x^2 + 6x + 8$
$y = x^2 - 6x - 16$	$y = (x - 8)(x + 2)$
$y = (x + 4)(x + 2)$	$y = (x + 2)(4 - x)$
$y = (x - 4)(2 - x)$	$y = (x - 8)(x - 2)$
$y = (x - 4)(x - 4)$	$y = (x + 8)(x - 2)$

https://aiminghigh.aimssec.ac.za/quadratic-matching-1/

https://aiminghigh.aimssec.ac.za/quadratic-matching-2/

NOTES FOR TEACHERS

SOLUTION

To solve the quadratic equation $ax^2+bx+c=0$ we rearrange it to complete the square.	Think about where the graph of $y=ax^2+bx+c$ cuts the y axis.	
$a(x+\frac{b}{2a})^2+c-\frac{b^2}{4a}=0$	Rearranging the equation.	
$a(x+\frac{b}{2a})^2-\frac{b^2-4ac}{4a}$ $=0$	The graph is a translation of $y=x^2$ by $-b/2a$ in the x direction and $\frac{b^2-4ac}{4a}$ in the negative y direction so it is symmetrical about the line $x=-b/2a$	
$(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$		
Next take the square roots of both sides $(x+\frac{b}{2a})=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$	The roots of the equation are given by the x coordinates of the points P and Q which are symmetric on either side of the line $x=-b/2a$	
The roots of the equation are $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	P = $(\frac{-b-\sqrt{b^2-4ac}}{2a},0)$ Q = $\frac{-b+\sqrt{b^2-4ac}}{2a},0)$	

Example

To solve $2x^2+9x-3=0$ by completing the square the equation can be rearranged to give

$$2\left[\left(x+\frac{9}{4}\right)^2-\frac{3}{2}-\frac{81}{16}\right]=0$$

$$2\left[\left(x+\frac{9}{4}\right)^2-\frac{105}{16}\right]=0$$

$$\left(x+\frac{9}{4}\right)^2=\frac{105}{16}$$

So the solutions are:

$x=-\frac{9}{4}\pm\frac{\sqrt{105}}{4}=\frac{-9\pm10.247}{4}$ = - 2.25 ± 2.562 = - 4.812 and 0.312 (to 3 decimal places)

Why do this activity?

This series of activities gives a well rounded and in-depth understanding of quadratic functions and quadratic equations.

Learning objectives

In doing this activity students will have an opportunity to:

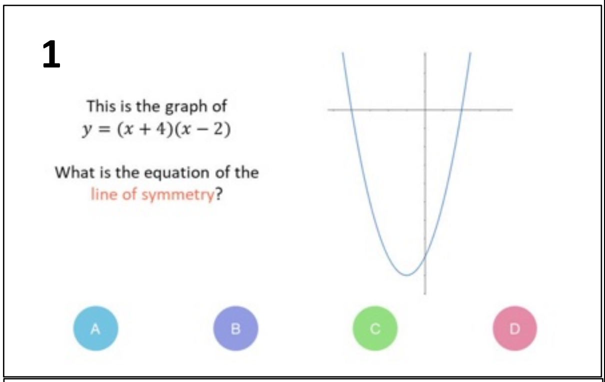
- put together a proof
- deepen their understanding of transformations of graphs
- connect the algebraic process of deriving the quadratic formula to transformations of graphs
- meet an introduction to complex numbers

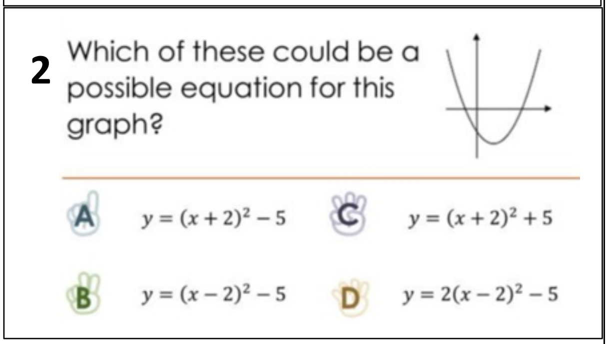
DIAGNOSTIC ASSESSMENT This should take about 5–10 minutes.

Write the question on the board, say to the class:

"Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 for D".

1. Notice how the learners respond. Ask a learner who gave answer A to explain why he or she gave that answer. DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
2. It is important for learners to explain the reasons for their answers. Putting thoughts into words may help them to gain better understanding and improve their communication skills.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. Ask the class to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.





Diagnostic Quiz **1** The correct answer is: D
Halfway between the roots x = -4 and x = 2.
Any other answer shows a serious lack of understanding of this topic.
Diagnostic Quiz **2** The correct answer is B
https://diagnosticquestions.com

Generic competences

In doing this activity students will have an opportunity to reason logically and to use powers of visualization.

Suggestions for teaching

This activity is designed for students to work through individually, or possibly in pairs.
The teacher might start with the two diagnostic quizzes and then support the students while they work through this series of activities.

Key questions

- What does the graph of the function corresponding to that equation look like?
- What transformations of the graph of $y = x^2$ have taken place to produce the graph of that function?
- How do the symmetry and other features of the graph of its related quadratic function relate to the formula for the solutions of a quadratic equation.

Follow up

Quadratic Functions https://aiminghigh.aimssec.ac.za/quadratic-functions/

Quadratic Matching 1 and 2 https://aiminghigh.aimssec.ac.za/quadratic-matching-1/
https://aiminghigh.aimssec.ac.za/quadratic-matching-2/

Go to the **AIMSSEC AIMING HIGH** website for lesson ideas, solutions and curriculum links: http://aiminghigh.aimssec.ac.za
Subscribe to the **MATHS TOYS YouTube Channel** https://www.youtube.com/c/MathsToys/videos
Download the whole AIMSSEC collection of resources to use offline with the **AIMSSEC App** see https://aimssec.app or find it on Google Play.

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and school years up to Secondary 5 in East Africa.
New material will be added for Secondary 6.
For resources for teaching A level mathematics (Years 12 and 13) see https://nrich.maths.org/12339
Mathematics taught in Year 13 (UK) & Secondary 6 (East Africa) is beyond the SA CAPS curriculum for Grade 12

	Lower Primary Approx. Age 5 to 8	Upper Primary Age 8 to 11	Lower Secondary Age 11 to 15	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13