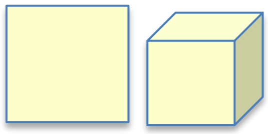
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

The HOW MANY FACTORS Inclusion and Home Learning Guide

is part of a Learning Pack downloadable from the AIMING HIGH website
https://aiminghigh.aimssec.ac.za/how_many_factors/
It provides related learning activities for all ages and learning stages from pre-school to school-leaving on the Common Theme **FACTORS** as a source for differentiation and inclusion in school lessons. There is guidance for home-learning and it is well suited to a mixed-age group who can do a starter together followed by age-appropriate learning activities.
Choose what seems suitable for the age or attainment level of your learners.

HOW MANY FACTORS? *Suitable for Years 7 to 11.*

See pages 3 -9 for related activities on the same topic for ages 4 to 18+.



How many factors does 72 have?
Investigate numbers that are products of a square number and a cube number. How many factors do they have?
How does the number 1 behave in the world of factors?
What can you say about numbers that have exactly 2 factors?
What can you say about numbers that have exactly 3 factors?
Give some examples of numbers with 4 factors. What do they have in common?
Give some examples of numbers with 5 factors. What do they have in common?
What about numbers with 6 factors?
Give some examples of numbers with 12 factors.
What is the smallest number with exactly fourteen divisors?

HELP

How many factors does 72 have?

Can you find a connection between the number of factors of 72 and the exponents in the expression giving the factorisation of 72 into primes?
$72 = 2^3 \times 3^2$.
You might find it easier to explore the problem in the context of finding all possible rectangles with whole number edge lengths that have the same area. For example you could draw all the rectangles with area 24 and you will find that there are four of them and they are: 1 by 24, 2 by 12, 3 by 8 and 4 by 6. Notice not 5 by anything and when you go up beyond 5 you just get the same rectangles again.

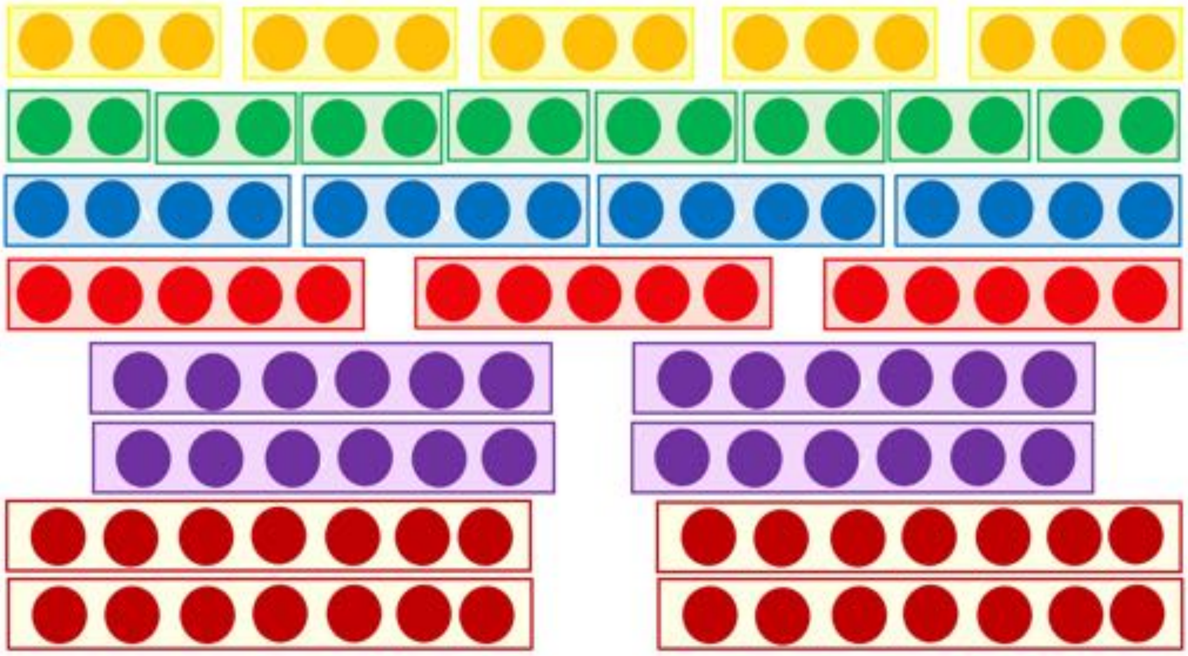
Then think about the problem of working out how many such rectangles there would be without drawing them all.

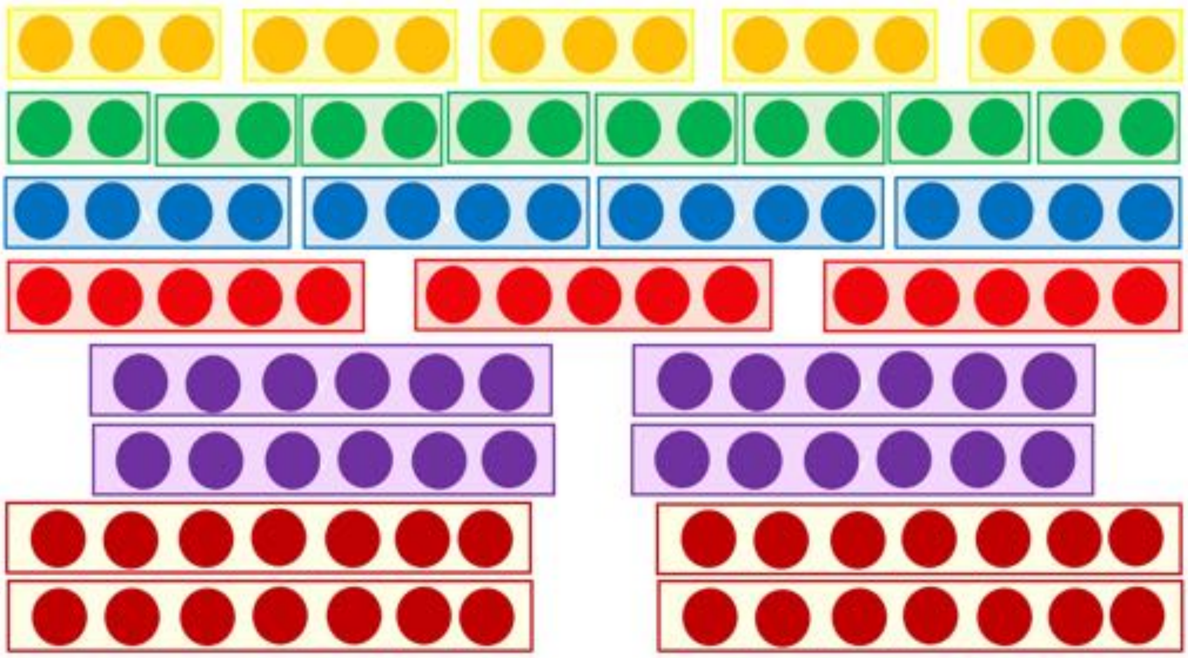
NEXT

1. When you know the prime factorisation of a number how do you find the TOTAL number of its factors?
2. What is the smallest number with exactly 100 factors?
3. Which number less than 1000 has the most factors?

These, and other similar questions, could be explored with paper and pencil using prime factorisation or it could be an opportunity for you to use a spreadsheet or simple coding (programming).

If you have an interest in programming you might wish to consider how to write a simple program to find all the factors of a number. For very large numbers, the realisation that you only need consider potential factors less than the square root of the number speeds up a program considerably!

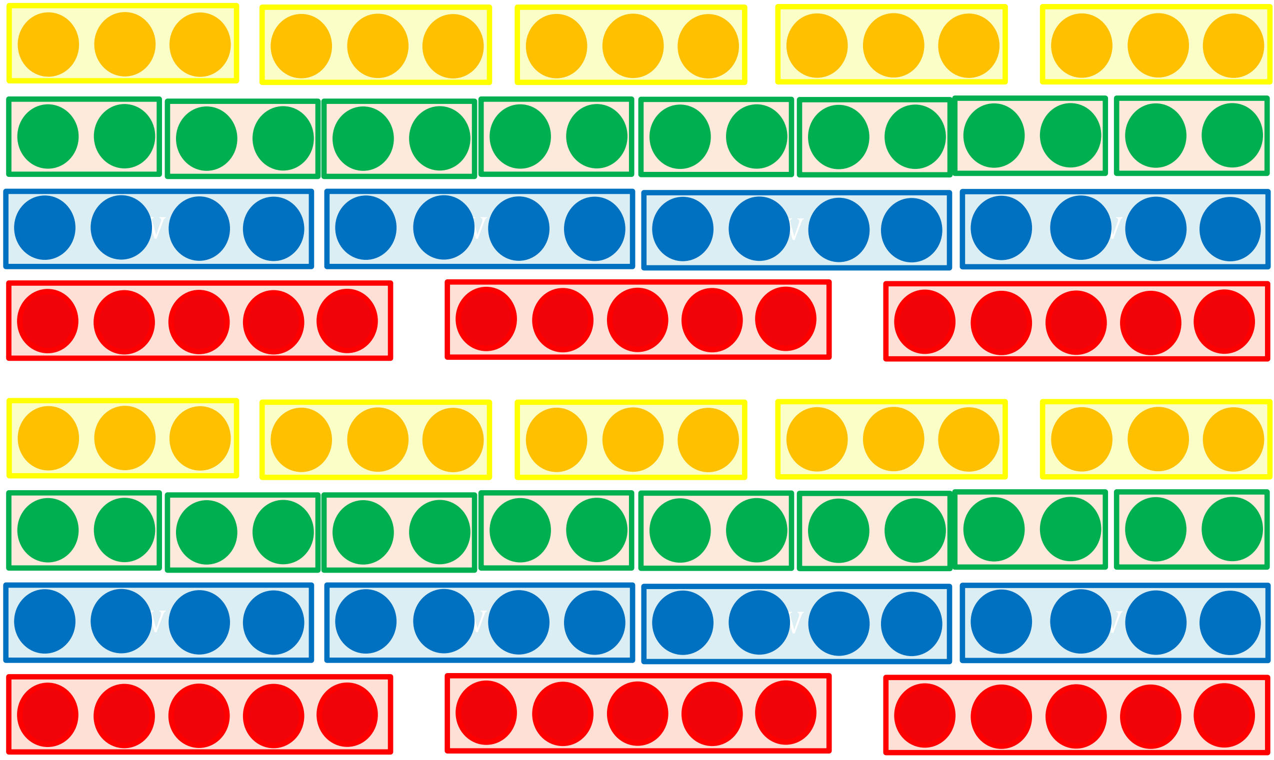




INCLUSION AND HOME LEARNING GUIDE
THEME: FACTORS

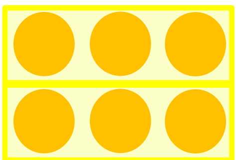
Early Years MAKING NUMBERS

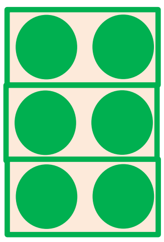
Cut out these pieces and use them to make rectangle patterns and numbers.



These two patterns make the number 6

Two times three make six   Three times two make six





What other numbers can you make in different ways?

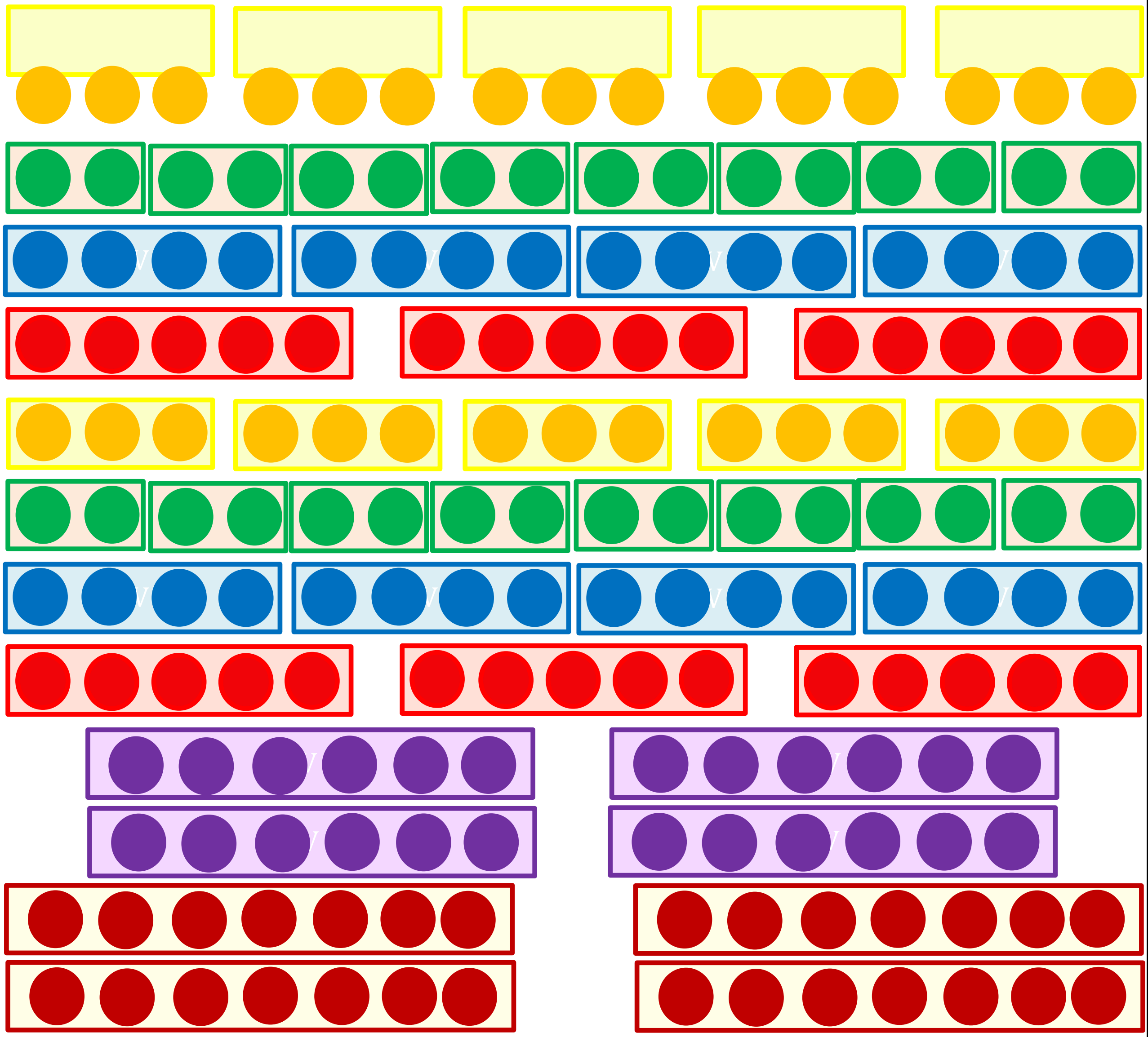
Note: Focus here on MULTIPLICATION*. Young children should concentrate on naming the sticks GREEN 2, Yellow 3, Blue 4 and Red 5 and on making rectangle patterns.*
Bear in mind that this work is laying the foundations for understanding calculations like:

$$2 \times 3 = 6 \qquad 3 \times 2 = 6$$
$$\text{and } 3 + 3 = 6 \qquad 2 + 2 + 2 = 6$$

and later learning multiplication tables and learning about factors and multiples.

Lower Primary

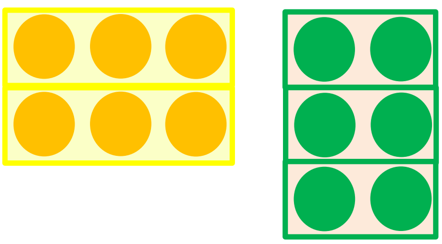
Cut out these pieces and use them to make rectangle patterns and numbers.



These two patterns make the number 6.

$2 \times 3 = 6$ $3 \times 2 = 6$

$3 + 3 = 6$ $2 + 2 + 2 = 6$



What other numbers can you make in different ways?

How many different ways can you make a rectangle for the number 12?

Note: Focus here on MULTIPLICATION. Bear in mind that this work is laying the foundations for learning multiplication tables and later understanding factors and multiples.

Upper Primary

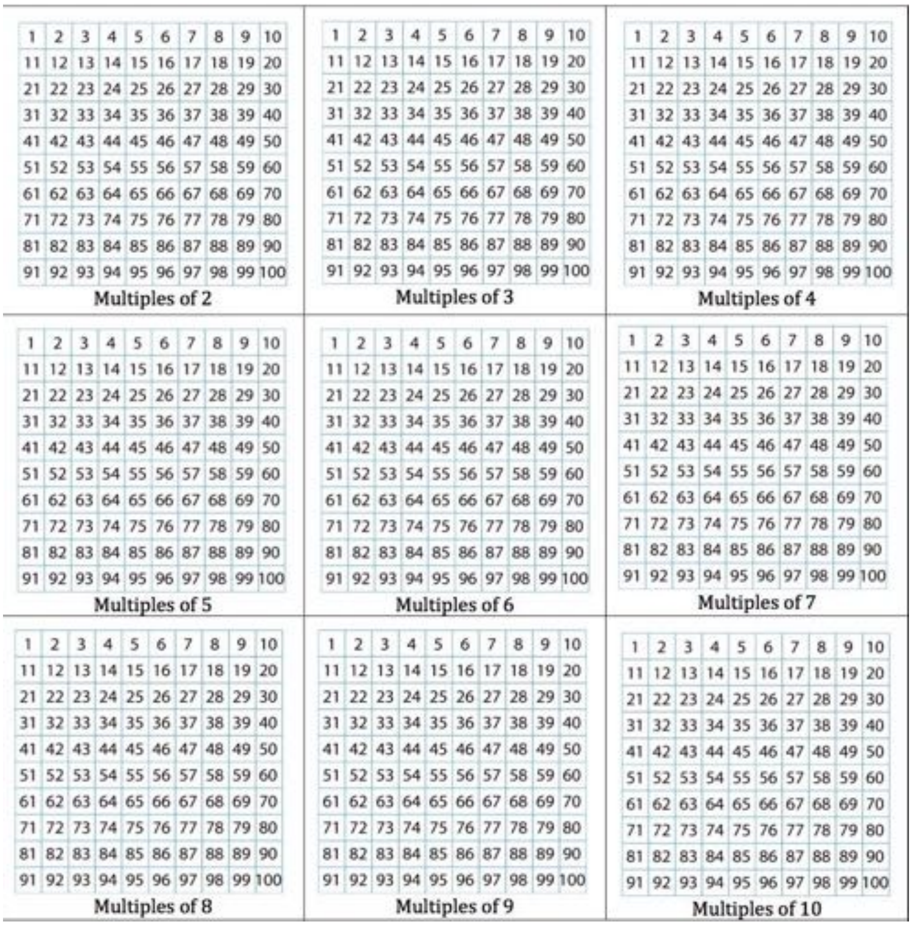
MULTIPLICATION TABLES, MULTIPLES AND FACTORS

As a starter cut out the pieces on page 3 and use them to make rectangle patterns and make the numbers 4, 6, 8, 10, 12 and 14 using only 2-sticks.

Which of these numbers can you make with 3-sticks? What about 5-sticks?

What numbers can you make with 5-sticks? What about 6-sticks? And 7-sticks?

***Note: Focus here on MULTIPLICATION.** Bear in mind that this work is laying the foundations for learning multiplication tables and understanding factors and multiples.*



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 2

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 3

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 4

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 5

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 6

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 7

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 8

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 9

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 10

With 10-year olds and older learner introduce the words factors and multiples and give them the Multiple-Patterns task.

Shade all the squares with multiples of the given number.

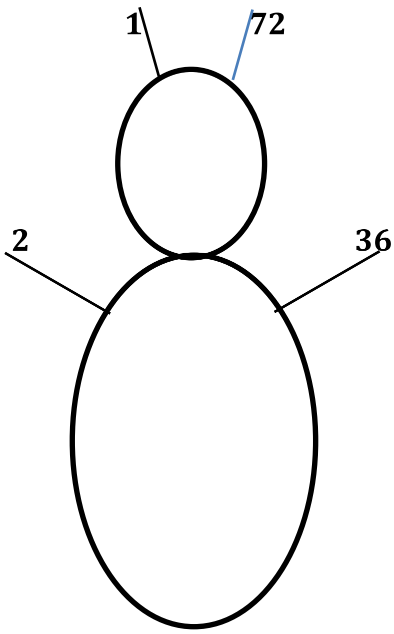
If this is done in groups in a lesson then each person could choose a different 100 square to colour in a set of multiples and the group could compare their different patterns.

See https://aiminghigh.aimssec.ac.za/multiple-patterns/activity

Learners could follow the Multiple Patterns with Prime Sieve.

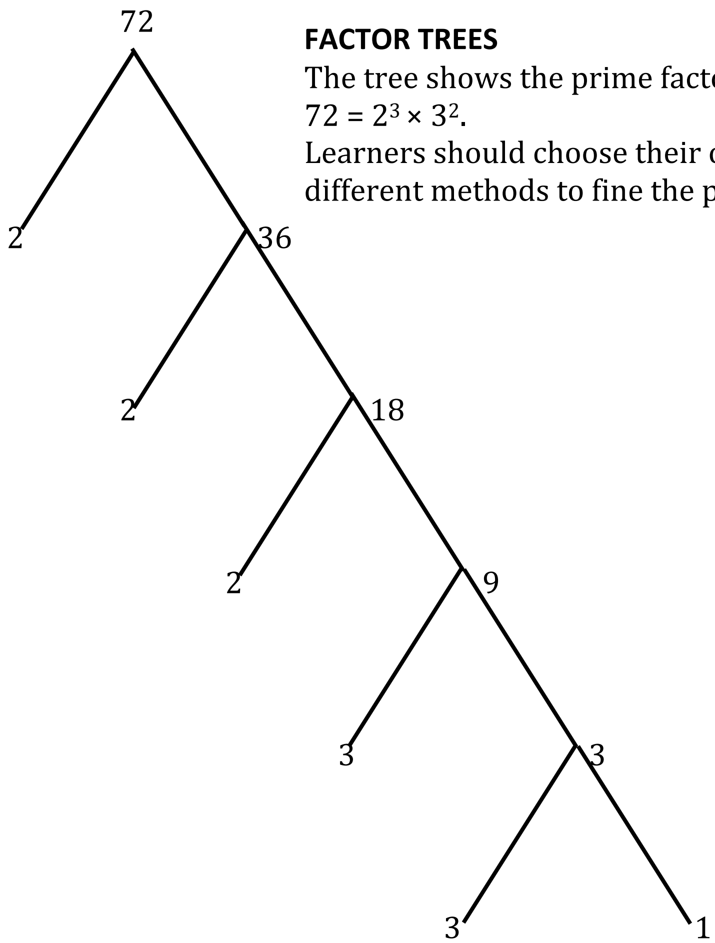
https://aiminghigh.aimssec.ac.za/prime-sieve/

Factor Bugs, Factor Trees and Factor Ladders are excellent tasks for the development of understanding of factors and multiples.



FACTOR BUGS

Draw pairs of legs to represent all the other pairs of factors of 72. **How many factors are there altogether?**



FACTOR TREES

The tree shows the prime factorization of $72 = 2^3 \times 3^2$.

Learners should choose their own numbers and use different methods to fine the prime factorization.

2	72
2	36
2	18
2	9
3	3
3	1

FACTOR LADDER
This is simply another way to record a method for finding the prime factorisation.
Learners should choose their own numbers and use different methods to fine the prime factorization.

Lower Secondary – Years 7 to 10 – See the activity on page 1

Start with the Diagnostic Quiz then ask learners to find the factors of 72.

Then ask your class to find all the factors of the numbers 1 to 40 and to indicate which factors are prime, and which are not.

TEAMWORK
Share the work so that different groups of learners find factors of different numbers.

Learners should write the factors that they have found in a table on the board to share their results with the class.

GROUP	Find factors of
1	1 – 5, 38, 39, 40
2	6 – 10, 35, 36, 37
3	11 – 15, 32, 33, 34
4	16 – 20, 29, 30, 31
5	21 – 28.

Ask the learners to record their results in a table of results as shown and to circle or highlight the prime numbers, square numbers and cube numbers on the table.

n	Number of factors	Factors
Primes numbers are highlighted in yellow Square numbers are highlighted in blue Cube numbers are highlighted in grey		
2	2	1, 2
3	2	1, 3
4	3	1, 2, 4
5	2	1, 5
6	4	1, 2, 3, 6
7	2	1, 7
8	4	1, 2, 4, 8
9	3	1, 3, 9
10	4	1, 2, 5, 10
11	2	1, 11
12	6	1, 2, 3, 4, 6, 12
13	2	1, 13
14	4	1, 2, 7, 14
15	4	1, 3, 5, 15
16	5	1, 2, 4, 8, 16
17	2	1, 17
18	6	1, 2, 3, 6, 9, 18
19	2	1, 19
20	6	1, 2, 4, 5, 10, 20
21	4	1, 3, 7, 21
22	4	1, 2, 11, 22
23	2	1, 23
24	8	1, 2, 3, 4, 6, 8, 12, 24
25	3	1, 5, 25
26	4	1, 2, 13, 26
27	4	1, 3, 9, 27
28	6	1, 2, 4, 7, 14, 28
29	2	1, 29
30	8	1, 2, 3, 5, 6, 10, 15, 30
31	2	1, 31
32	6	1, 2, 4, 8, 16, 32
33	4	1, 3, 11, 33
34	4	1, 2, 17, 34
35	4	1, 5, 7, 35
36	9	1, 2, 3, 4, 6, 9, 12, 18, 36
37	2	1, 37
38	4	1, 2, 19, 38
39	4	1, 3, 13, 39
40	8	1, 2, 4, 5, 8, 10, 20, 40

Then either write the questions from page 1 on the board or give out copies of the worksheet. Suggest that their discoveries about the factors of the numbers 1 to 40 will help them to do the problem.

Ask learners to work on the problem individually for about 10 minutes and after that to share their work in pairs. While learners are working observe what they are doing, asking key questions to guide them when necessary but NOT telling them what they should do.

Most learners will probably list all the factor pairs but the question "What can you say about numbers that have exactly 2 factors?" should get them thinking about prime numbers. Once they have had a go, share strategies and answers as a whole group, encouraging learners to explain their working.

You might introduce a discussion of security codes for financial transactions. The lock on the code protecting your private information is a very big number which is the unique product of two prime numbers, and the key to this lock are exactly these two primes. For a very large number it would take a lot of checking to find the two "key" primes if you don't already know them.

The best computers in the world might take years to find the “key” primes of a “lock” number only 300 digits long. In the future faster computers will be able to factorise much bigger numbers quickly and we shall need better security codes.

Finally summarize what has been learned.

Diagnostic Assessment This should take about 5–10 minutes.

Write the question on the board, say to the class:

“Put up 1 finger if you think the answer is A, 2 fingers for B, 3 for C, and 4 for D”.

Which of these is 72 written as a product of primes?

A $2 \times 2 \times 2 \times 2 \times 3 \times 3$

B $2 \times 2 \times 2 \times 9$

C $2^4 \times 3^2$

D $2^3 \times 3^2$

1. Notice how the learners respond. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
2. Learners should explain the reason for their answers to practise communication skills and to develop their thinking by having to express their thoughts in words.
3. Do the same for answers B, C and D. Make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.**
5. The concept is needed for the lesson so explain it or give a remedial task.

The correct answer is **D**: $72 = 8 \times 9 = 2^3 \times 3^2$

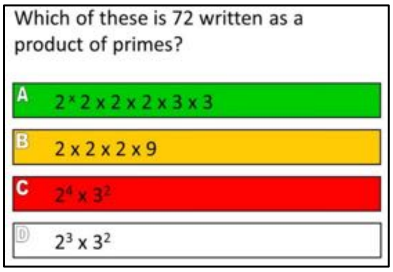
A. Wrong on the factors – this is 16×9

B. 9 is not a prime number

C. Wrong on the factors again– this is 16×9

https://diagnosticquestions.

Upper Secondary



Start with this Diagnostic Assessment.

Do the learning activity on pages 1 and 2. This is a non-standard problem. First investigate prime factorisation and writing numbers as products of primes and finding all the factors of numbers.

When you investigate types of numbers according to how many factors they have it will be helpful to list all the factors of some numbers, say from 1 to 40, to identify prime, square and cube numbers and to look for patterns in the numbers of factors. You need to work systematically.

There is a connection with probability where we need to include all possible events.

Factors of 360

Powers of 2	Powers of 3	Powers of 5	Powers of 2 × powers of 3 × powers of 5
2^0	3^0	5^0	1
		5^1	5
	3^1	5^0	3
		5^1	15
	3^2	5^0	9
		5^1	45
2^1	3^0	5^0	2
		5^1	10
	3^1	5^0	6
		5^1	30
	3^2	5^0	18
		5^1	90
2^2	3^0	5^0	4
		5^1	20
	3^1	5^0	12
		5^1	60
	3^2	5^0	36
		5^1	180
2^3	3^0	5^0	8
		5^1	40
	3^1	5^0	24
		5^1	120
	3^2	5^0	72
		5^1 →	360

Finding the prime factorisation does not immediately give you all the factors including factors that are not prime. For example, $360 = 2^3 \times 3^2 \times 5$. The table shows a systematic method for finding all the factors of 360 by listing all the numbers that are combinations of powers of 2^0 or 2 or 2^2 or 2^3 and 3^0 or 3 or 3^2 and 5^0 or 5.

Imagine a tree diagram presented in the table, with 4 branches, then 4 sets of 3 branches, then 12 sets of 2 branches.

The number 360 has 3 prime factors and 24 factors altogether.

You could also investigate the application of prime factorisation to security codes for online banking transactions.

If you can write computer programs you will find that it is easy to write the code to compute a list of numbers and their prime factors. The code below in Octave can easily be adapted to different languages. For very large numbers, the realisation that you only need to consider potential factors less than the square root of the number speeds up a program considerably.

```
for n = (1:40)
O = [1];
S=create_set(O);        these two lines create the set S = {1}
c=0;                    this sets the counter c at 0
for d=(1:n)
if floor(n/d)==n/d      this looks at all divisors d of n
c=c+1;                  this increases c by 1
y=[d];
S = union(S,y);         this adds d to the set S
end end
x = [n,c,S];
disp(x)                 this prints the numbers and the number of factors and list the set of factors
end
```

The **Key Questions** below should guide you in this work.

Key Questions

- What is special about the numbers you are factorising?
- How do you find a number with an odd number of factors? Find some examples?
- Do factors always come in pairs?
- If you have found a number there with 6 factors; can you find any other numbers with 6 factors?
- Check that 12 and 18 and 20 have 6 factors. What is the same and what is different about their factors?

Why do this activity?

This activity invites learners to tackle a non-standard problem and to explore the use of a prime factorisation representation of a number. Here learners have a context in which they need to work systematically. This activity can also lead to discussion of the application of prime factorisation to security codes for online banking transactions.

Learning objectives

In doing this activity students will have an opportunity to review and deepen their understanding of factors and prime factorization.

Generic competences

In doing this activity students will have an opportunity to:

- think mathematically, reason logically and give explanations and proofs;
- communicate in writing, speaking and listening according to the audience:
    - exchange ideas, criticise, and present information and ideas to others
    - analyze, reason and record ideas effectively.

SOLUTION

All prime numbers have exactly **2 factors**, the number itself and 1. For example the factors of 7 are 1 and 7.

All square numbers have exactly **3 factors**. For example $9 = 3^2$ has factors 1, 3 and 9.

The number $8 = 2^3$ has **4 factors** 1, 2, 4 and 8 and the number $10 = 2 \times 5$ has 4 factors 1, 2, 5 and 10.

Numbers with 4 factors are either the cube of a prime or the product of 2 primes.

The number $16= 2^4$ has **5 factors** 1, 2, 4, 8, 16. All numbers with 5 factors are the 4th power of a prime.

The number $32= 2^5$ has **6 factors** 1, 2, 4, 8, 16, 32. Also $12 = 2^2 \times 3$ has 6 factors 1, 2, 3, 4, 6, 12.

All numbers with 6 factors are either the 5th power of a prime number or a product of a square number and a cube number.

$72 = 2^3 \times 3^2$ has **12 factors** 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.

The powers of 2 that occur in the factors of 72 there are 2^0, 2^1, 2^2 and 2^3

and the powers of 3 that occur are 3^0, 3^1, and 3^2.

So the factors of 72 can contain powers of 2 in 4 ways and powers of 3 in 3 ways so there are 4 × 3 different factors of 72:

$2^0 3^0$, $2^0 3^1$, $2^0 3^2$,

$2^1 3^0$, $2^1 3^1$, $2^1 3^2$

$2^2 3^0$, $2^2 3^1$, $2^2 3^2$

$2^3 3^0$, $2^3 3^1$, $2^3 3^2$.

Similarly $108 = 2^2 \times 3^3$ has 12 factors 1, 2, 3, 4, 6, 9, 12, 18 27, 36, 54, 108. Also 2048 $= 2^{11}$ has 12 factors. But the smallest number with 12 factors is 72 because the numbers will either be of the form a^2b^3 or a^{11} where a and b are primes, so to find the smallest we take a=3 and b=2.

Numbers with 14 factors are either of the form a^{13} or a^1b^6 so the smallest number with 14 factors is

$3 \times 2^6 = 192$ which has factors

$2^0 3^0 = 1$, $2^1 3^0 = 2$, $2^2 3^0 = 4$, $2^3 3^0 = 8$, $2^4 3^0 = 16$, $2^5 3^0 = 32$, $2^6 3^0 = 64$,

$2^0 3^1 = 3$, $2^1 3^1 = 6$, $2^2 3^1 = 12$, $2^3 3^1 = 24$, $2^4 3^1 = 48$, $2^5 3^1 = 96$, $2^6 3^1 = 192$.

Amongst the other numbers with 14 factors are $2^{13} = 8192$ and $2^1 \times 3^6 = 1458$.

NEXT

1. **When you know the prime factorisation of a number how do you find the TOTAL number of its factors?**
2. **What is the smallest number with exactly 100 factors?**
3. **Which number less than 1000 has the most factors?**

The solution uses the **fundamental counting principle** which is a rule used to count the total number of possible outcomes in a situation.

If there are m ways of doing something, and n ways of doing another thing after that, then there are m $\times$ n ways to perform both actions.

For the number of ways to perform 3 or more actions, multiply the number of ways of doing each action.

1. For each ${a_r}^{p_r}$ in the prime factorisation, when you count all the possibilities,

a_r contributes in (p_r+ 1) different ways to the factors,

that is either ${a_r}^0$ or ${a_r}^1$ or ${a_r}^2$ or ... ${a_r}^{p_r}$, altogether (p_r+ 1) different possibilities.

So a number with prime factorisation

${a_1}^{p_1} \times {a_2}^{p_2} \times {a_3}^{p_3} \times \ldots \times {a_r}^{p_r} \times \ldots \times {a_n}^{p_n}$

has $(p_1+1)(p_2+1)(p_3+1)\ldots(p_n+1)$ factors.

2. 2^{99} is the smallest number with exactly 100 factors.

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3. To find the number less than 1000 that has the most factors, you have to try and multiply together the widest range of prime numbers, starting with the lowest:
   $2 \times 3 \times 5 \times 7 \times 11 > 1000$ but,
   $2 \times 3 \times 5 \times 7 < 1000$ so these are the numbers that we will use.

$2 \times 3 \times 5 \times 7 = 210$, so, we can use powers of 2 until the number is as close to 1000 as possible. The number closest to 1000 that has the most factors is:
$2^3 \times 3^1 \times 5^1 \times 7^1 = 840$
The number 840 has $4 \times 2 \times 2 \times 2 = 32$ factors.

Follow-up ideas

Ishango Bone https://aiminghigh.aimssec.ac.za/years-6-12-ishango-bone/
Who am I? https://aiminghigh.aimssec.ac.za/years-7-to-9-who-am-i/
Prime Seive https://aiminghigh.aimssec.ac.za/years-6-9-prime-sieve/
Factors and Multiple Game
https://aiminghigh.aimssec.ac.za/years-6-12-factors-and-multiples-game/