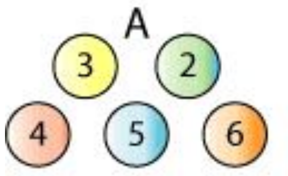
ODDS AND EVENS



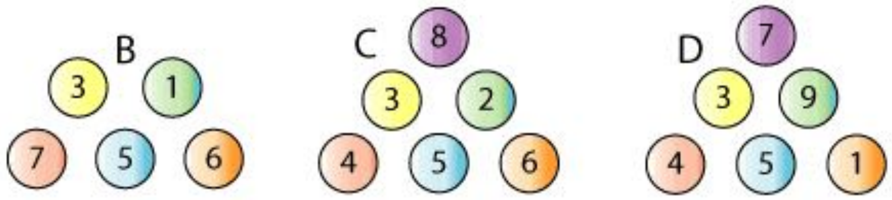
In Game A you pick out two balls at random from a bag containing 5 balls numbered 2, 3, 4, 5 and 6.

If the total is even you win. If it is odd you lose.

Are you equally likely to win or to lose (a fair game)?

Games B, C and D use the numbers shown below and the rules are the same.

Which set of balls would you choose to give yourself the best chance of winning?



Do your own experiments. Make number cards and put them in an envelope or bag.

To do interactive computer simulations of these games see http://nrich.maths.org/4308

HELP

	First choice					
Second choice	A	2	3	4	5	6
	2	X			7	
	3		X		8	
	4	6		X		
	5			9	X	
	6					X

Play Game A a few times. You are conducting trials in a probability experiment.

Then fill in a table as shown colouring the odd and even totals in contrasting colours and putting in crosses because you can't choose the same ball twice.

This is called **a sample space diagram**.

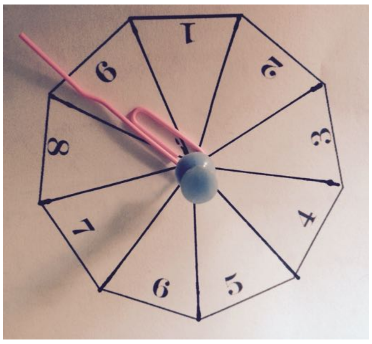
The sample space diagram shows that there are exactly 20 possible outcomes when you choose 2 cards at random, that is exactly 20 elements in the sample space.

NEXT

None of the sets looked at so far (A, B, C and D) gives a fair game.
Can you find out whether there are any sets that would give a fair game?

1	**2**	**3**
4	**5**	**6**
7	**8**	**9**

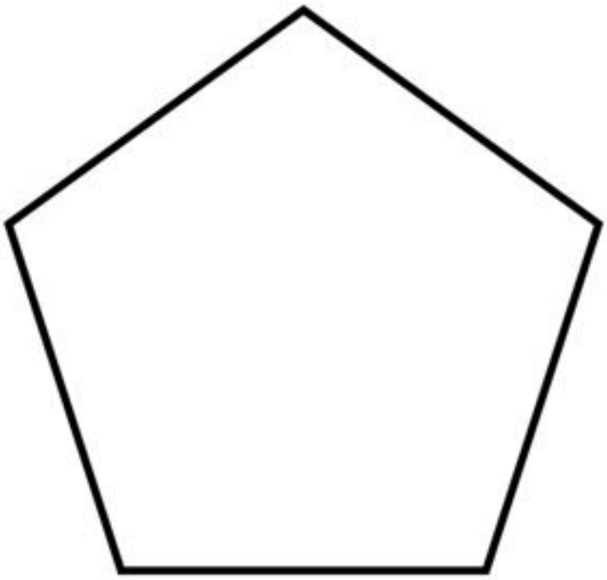
MAKING SPINNERS FOR THESE GAMES

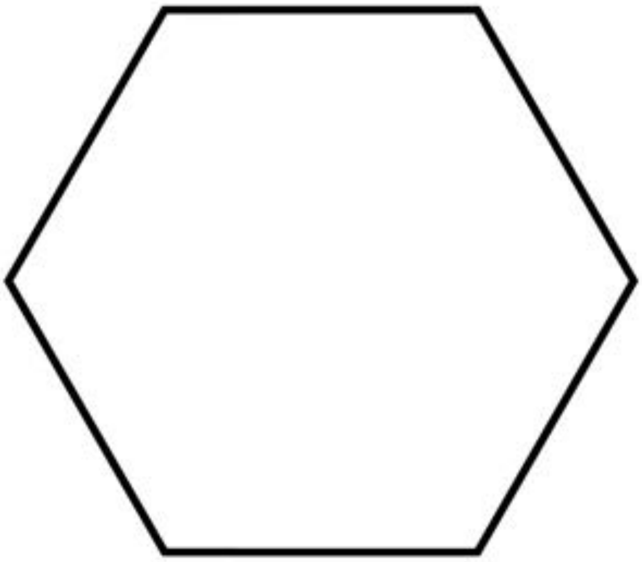


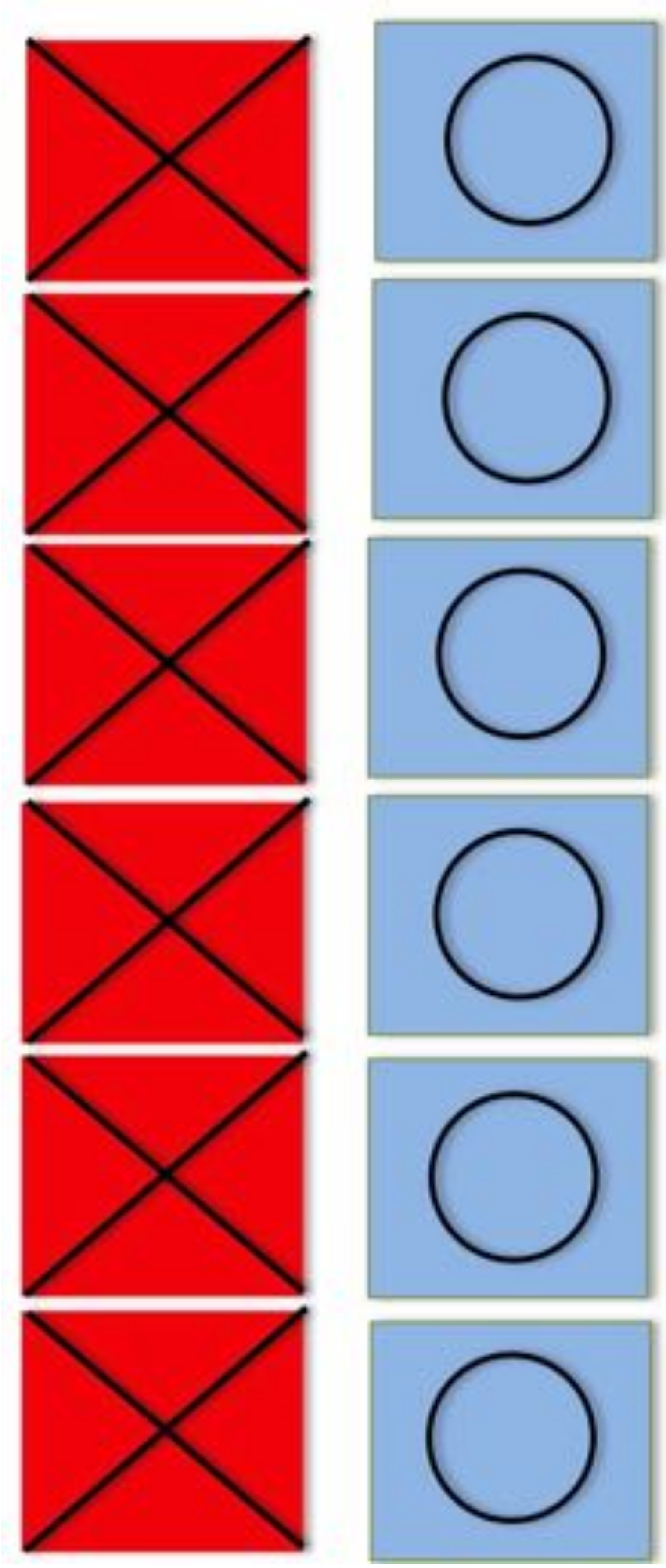
You can simulate the games using spinners made from the templates below, a pin and a paperclip.

Mark the segments on the spinners, and the 5 or 6 numbers for the game.

Cut out and pin on a flat surface through the paper clip and the template (as in the picture) making sure that the paper clip spins freely.







PROOF SORTING EXERCISE FOR YEAR 12 AND 13 STUDENTS.

This is the 1st statement. Cut out the strips and re-construct the proof by arranging the remaining statements in the correct order.

Odds and Evens

Imagine we play the game with a even numbered balls and b odd numbered balls.

If a and b are any two consecutive triangle numbers, the reverse argument holds.
We have proved that a necessary and sufficient condition for a game of this type to be a fair game is that the total number of balls ($a + b$) is a square number n^2, moreover a and b turn out to be consecutive triangle numbers.

For a fair game we must have the probability of an even sum equal to the probability of an odd sum; that is:

$(a^2 - a) + (b^2 - b) = ab + ab$

$\Leftrightarrow \quad a^2 - 2ab + b^2 = a + b$

$\Leftrightarrow \qquad (a - b)^2 = a + b$

This suggests that a and b are consecutive triangle numbers n because the rule for generating the sequence of triangle numbers is that the difference between the n^{th} and $(n-1)^{th}$ number is equal to n.

Proof:

For a even and b odd numbers this is the sample space diagram, with blue for even totals and yellow for odd totals:

	a even	b odd
a even	$a^2 - a$	ab
b odd	ab	$b^2 - b$

We know that

$a + b = n^2$

and $a - b = n$ (or $b - a = n$ if $b > a$).

We have proved that: the game will be a fair game if and only if $(a - b)^2 = a + b$

This leads us to think that:

- the total number of balls (i.e. $a + b$) has to be a square number, say n^2;
- the difference between the number of odd and even balls is n.

Adding the two equations:

$2a = n^2 + n$

=> $a = ½ (n^2 + n)$

=> $a = ½n (n+1)$

This formula gives a as the n^{th} triangular number.

Similarly, b is the $(n - 1)^{th}$ triangular number $½n (n - 1)$.

Conjecture: The game will be a fair game if and only if the number of balls is a square number, in which case the number of odd, and even balls, a and b, are two consecutive triangle numbers.

NOTES FOR TEACHERS

For learning activities for different ages and attainment levels see the Inclusion and Home Learning Guide.

SOLUTION

Game A

Probability of an even sum is $\frac{8}{20}=\frac{2}{5}$.
Probability of an odd sum is $\frac{12}{20}=\frac{3}{5}$.
So it is not a fair game.

	2	3	4	5	6
2	X	5	6	7	8
3	5	X	7	8	9
4	6	7	X	9	10
5	7	8	9	X	11
6	8	9	10	11	X

Game B

Probability of an even sum is $\frac{12}{20}=\frac{3}{5}$.
Probability of an odd sum is $\frac{8}{20}=\frac{2}{5}$.
So it is not a fair game.

	1	3	5	6	7
1	X	4	6	7	8
3	4	X	8	9	10
5	6	8	X	11	12
6	7	9	11	X	13
7	8	10	12	13	X

Game C

Probability of an even sum is $\frac{14}{30}=\frac{7}{15}$.
Probability of an odd sum is $\frac{16}{30}=\frac{8}{15}$.
So it is not a fair game.

	2	3	4	5	6	8
2	X	5	6	7	8	18
3	5	X	7	8	9	11
4	6	7	X	9	10	12
5	7	8	9	X	11	13
6	8	9	10	11	X	14
8	10	11	12	13	14	X

Game D

Probability of an even sum is $\frac{21}{30}=\frac{7}{10}$.
Probability of an odd sum is $\frac{9}{30}=\frac{3}{10}$.
So it is not a fair game.

	1	3	4	5	7	9
1	X	4	5	6	8	10
3	4	X	7	8	10	12
4	5	8	X	9	11	13
5	6	8	9	X	12	18
7	8	10	11	12	X	16
9	10	12	13	14	16	X

See page 11 for a proof of the conditions for a game of this type to be a fair game.

Why do this activity?

In this activity learners can explore and discuss two types of probability: experimental and theoretical. They can experience teamwork in trying to find an example of a fair game of this type and trying to formulate a conjectures about what conditions might lead to a fair game. Older learners can work on the challenge of proving the conjectures.

Learning objectives

In doing this activity students will have an opportunity to:

- conduct probability experiments and record results;
- experience and discuss random sampling;
- learn to use a sample space diagram to show all the elements in a sample space and to analyse the theoretical probabilities of different events;
- reflect on the differences between experimental probability and theoretical probability;
- find and prove conjectures about conditions for a fair game.

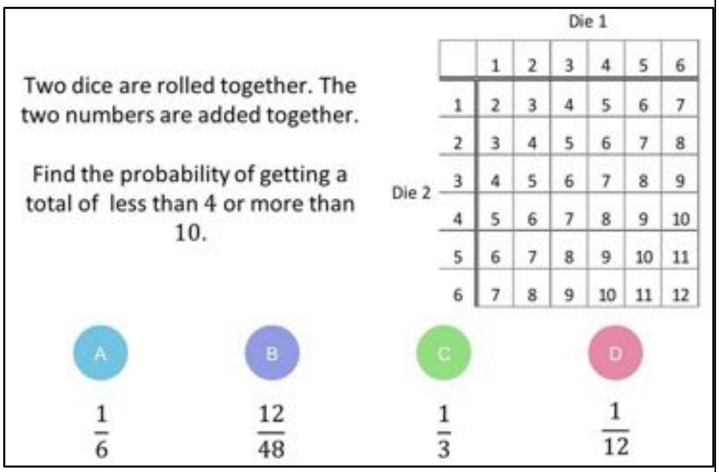
Generic competences

In doing this activity students will have an opportunity to:

- conduct probability experiments, reflect on the differences between experimental probability and theoretical probability and discuss the types of applications that require the use of one or the other;
- work as a team towards a common goal;
- **communicate** in writing, speaking and listening:
    - exchange ideas, criticise, and present information and ideas to others;
    - analyze, reason and record ideas effectively.

Diagnostic Assessment This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 for C and 4 for D".



2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. It is important for learners to explain the reason for their answer to clarify their own thinking and to practise communication skills.
4. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
5. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.**
6. The concept is needed for the lesson to follow, so explain the right answer or give a remedial task.

   A. is the correct answer. There are 6 possible outcomes out of 36.

Common Misconceptions

B. Maybe a guess

C. These learners might have included scores of 4 and 10. Could be a misunderstanding about 'less than'.

D. Maybe a guess https://diagnosticquestions.com

Suggestions for teaching Years 9 to 12

For learning activities for younger learners including pre-school and primary years, see the Inclusion and Home Learning Guide.

Resources *Learners can simulate the games using numbered cards (see page 2) or spinners (see page 3) or using numbered counters in a bag.*

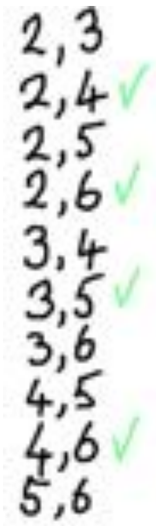
Start with the diagnostic question so that learners will then have sample space diagrams (2-way tables) in mind when they tackle the activity.

The notes that follow are based on lessons with a group of 14 year old students. The notes are in two parts: the first part for teachers who wish to use the activity for a single lesson on probability and sample space diagrams or tree diagrams. The second part is for teachers who wish to follow this up with a collaborative task that leads to interesting and unexpected results, and also for teachers of older students.

LESSON 1 Start by showing how the game is played using Set A numbered cards in a bag or using the computer simulation http://nrich.maths.org/4308. The computer simulation generates lots of experimental data quickly, freeing time to focus on predictions, analysis and justifications. Calculating the theoretical probabilities provides a motivation for using sample space diagrams or perhaps tree diagrams.

Play the game no more than ten times, so that learners have a feel for the game but don't have sufficient results to draw conclusions about the probabilities. Then ask them to decide whether they think the game is fair, and to do some maths to support their decision.

While students are working, circulate and observe the methods being used:



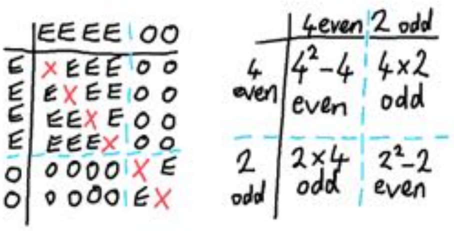
Bring the class together and choose individuals who used different methods to explain what they did to the class, recording what they did on the board. Perhaps choose those who used less sophisticated methods first. To prepare students for tackling other examples with a large number of possibilities, emphasise the merits of a sample space method rather than a listing method. Those who are confident with tree diagrams may prefer to continue using them.

If you have the internet, use the computer interactivity a large number of times to confirm that the experimental probability matches closely to the theoretical probability that students have calculated. There are opportunities here for rich discussion about how closely we expect an experimental probability to match the theory.

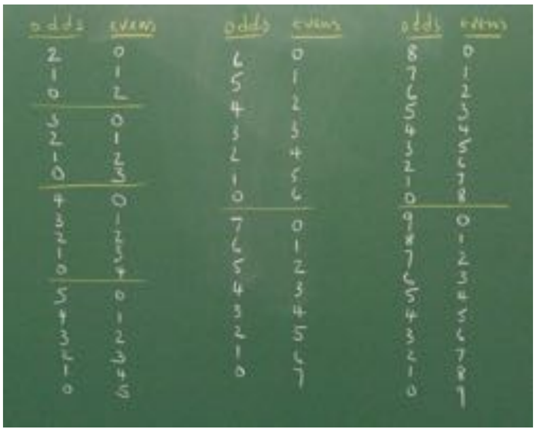
HOMEWORK OR FOLLOW-ON QUESTION FOR THE NEXT LESSON:

Now show sets B, C and D, and ask the students to **think on their own, without writing,** about which of the four sets they would choose to play with, to maximise their chances of winning and to work out the respective probabilities. Next lesson, once the students have had a short time to reflect, ask them **to work in pairs** to discuss their choices, and to justify their decisions (again, without writing). There is often disagreement about which set offers the best chance of winning, so **bring the class together to compare ideas**. Discourage the use of inefficient listing methods.

Lesson 2: Once the probabilities have been calculated for Games, A, B, C and D, you might use the interactivity again to confirm that the experimental probability is close to the calculated one.
Now write up on the board a set E, which contains four large even numbers and two large odd numbers. Make them large enough that calculations would be off-putting! Ask the class to work in pairs to calculate the probability of winning with set E, and give them a short time frame in which to do this. The intention is to alert students that the numbers themselves don't matter, but the numbers of odds and evens is the important point. Set E has the same structure as Set C, so we already know the chance of winning. Then the class can be introduced to this sort of sample space diagram where odds and evens are collected together:



None of the sets looked at so far (A, B, C, D and E) gives a fair game.
Ask "How could we go about finding out whether there are any sets that would give a fair game?"



One way of organising the search is to draw up a table on the board showing different combinations of odds and evens.
Those already identified as not being fair games (sets A, B, C and D) can be crossed off. Then divide the class into groups working on different combinations and ask them to report back. Students could record combinations that have been checked on the board with a tick or a cross to show whether they are fair or not. If something has two ticks or two crosses, it could be accepted as being confirmed. When disagreements arise, ask other groups to resolve them.

There will be opportunities while the class are working to stop everyone and share students' insights that will make the job easier.

For example:
"None of the combinations with zero will work because..."
"If 3 odds and 2 evens won't work, 2 odds and 3 evens won't either, because..."
"You can't have the same number of evens and odds because..."

Eventually, there will be a sea of crosses on the board and just a few combinations that work (four, if the class have gone up to 9 balls in total). Ask the class to stop and consider what the fair sets have in common. This may lead to some new conjectures about the total number of balls, so organise the class to test the conjecture out on the next obvious total.

Once there is some confirmation about the total number of balls needed for fair games, conjectures can also be made about how these should be split into odds and evens. Students can be set to work to test examples with large numbers, using the simplified sample space method above. Draw attention to how valuable it is to work collaboratively as part of a mathematical community, and how difficult it would have been to have reached the same insights working alone.

Key questions

- How can you decide if a game is fair?
- What are the most efficient methods for recording possible combinations?
- How can we make this difficult task (of finding a fair game) more manageable?

Follow up

The activity **In a Box** https://aiminghigh.aimssec.ac.za/years-6-12-in-a-box/ offers another context for exploring exactly the same underlying mathematical structure. Use it as a follow-up problem a few weeks after working on Odds and Evens.

Also see **Special Sums** https://aiminghigh.aimssec.ac.za/years-6-10-special-sums/

Prove your conjectures about the conditions for a fair game.

Although it is unlikely that many students will be able to prove their conjectures algebraically on their own, the following proof is sufficiently accessible to be worth sharing with at least some learners in a class of 14 to 16 year olds and for students in the last two years of school mathematics. There are several ways to use this resource:

- Present it as an elegant way of proving the ideas the learners have discovered.
- Use it as a **'proof sorting' exercise** where the proof is cut into sections and mixed up for students to reassemble into the correct order (see page 5).
- Present the proof on the board and organise a class discussion about it with a Q&A session. Then erase it and ask students to recreate it for themselves.
- Print out the proof on page 12, and distribute it to students for them to make sense of, and for them to annotate so that they could talk through the proof, line by line and explain it to someone who hadn't met it yet.

Odds and Evens Games

Imagine we play the game with *a* even numbered balls and *b* odd numbered balls.

Conjecture: The game will be a fair game if and only if the number of balls is a square number, in which case the number of odd, and even balls, *a* and *b,* are two consecutive triangle numbers.

Proof:

For *a* even and b odd numbers this is the sample space diagram, with blue for even totals and yellow for odd totals:

	a even	*b* odd
a even	$a^2 - a$	ab
b odd	ab	$b^2 - b$

For a fair game we must have the probability of an even sum equal to the probability of an odd sum; that is:

$(a^2 - a) + (b^2 - b) = ab + ab$

$\Leftrightarrow \; a^2 - 2ab + b^2 = a + b$

$\Leftrightarrow \quad (a - b)^2 = a + b$

We have proved that: the game will be a fair game if and only if $(a - b)^2 = a + b$

This leads us to think that:

- the total number of balls (i.e. $a + b$) has to be a square number, say n^2;
- the difference between the number of odd and even balls is n.

This suggests that *a* and *b* are consecutive triangle numbers n because the rule for generating the sequence of triangle numbers is that the difference between the n^{th} and $(n\text{-}1)^{th}$ number is equal to n.

We know that

$a + b = n^2$

and $a - b = n$ (or $b - a = n$ if $b > a$).

Adding the two equations:

$2a = n^2 + n$

$\Rightarrow a = ½ (n^2 + n)$

$\Rightarrow a = ½n (n+1)$

This formula gives *a* as the n^{th} triangular number.

Similarly, *b* is the $(n - 1)^{th}$ triangular number $½n (n - 1)$.

If a and b are any two consecutive triangle numbers, the reverse argument holds.

We have proved that a necessary and sufficient condition for a game of this type to be a fair game is that the total number of balls $(a + b)$ is a square number n^2, moreover *a* and *b* turn out to be consecutive triangle numbers.

An isomorphic game with balls of 2 different colours (a red balls and b green balls) is a fair game with a win being picking two balls of the same colour.