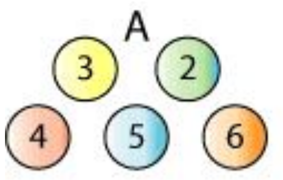
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

The ODDS AND EVENS Inclusion and Home Learning Guide provides related activities for lessons in school and home learning for all ages and learning stages from pre-school to school-leaving, on the Common Theme PROBABILITY SAMPLE SPACES

Choose what seems suitable for the age or attainment level of your learners.

https://aiminghigh.aimssec.ac.za/odds-and-evens/

ODDS AND EVENS



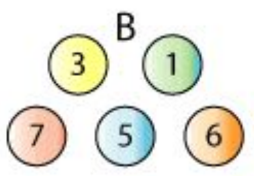
In Game A you pick out two balls at random from a bag containing 5 balls numbered 2, 3, 4, 5 and 6.

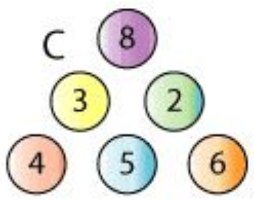
If the total is even you win. If it is odd you lose.

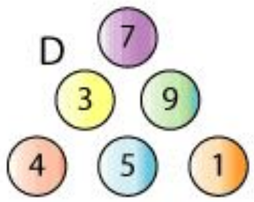
Are you equally likely to win or to lose (a fair game)?

Games B, C and D use the numbers shown below and the rules are the same.

Which set of balls would you choose to give yourself the best chance of winning?







Do your own experiments. Make number cards and put them in an envelope or bag.

For interactive computer simulations of the games see http://nrich.maths.org/4308

HELP

	First choice					
Second choice	A	2	3	5	4	6
	2	X		7		
	3		X	8		
	5	7		X		
	4			9	X	
	6					X

Play Game A several times and record the results. You are conducting trials in a probability experiment.

Then fill in a table as shown colouring the odd and even totals in contrasting colours and putting in crosses because you can't choose the same ball twice.

This is called **a sample space diagram**.

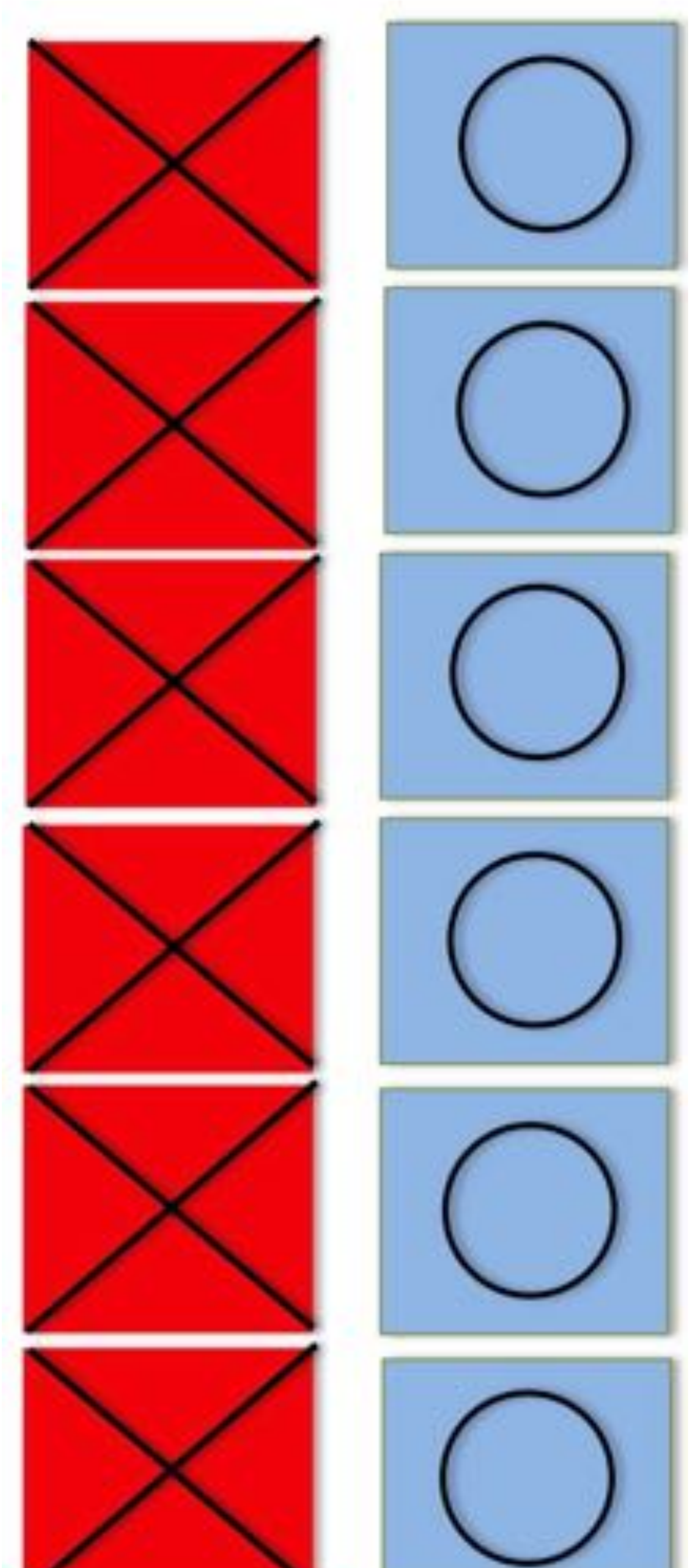
The sample space diagram shows that there are exactly 20 possible outcomes when you choose 2 cards at random. There are 20 elements in the sample space.

NEXT

None of the sets looked at so far (A, B, C and D) gives a fair game.

Can you find a set of numbers that would give a fair game?

1	2	3
4	5	6
7	8	9



PROOF SORTING EXERCISE FOR YEAR 12 AND 13 STUDENTS.

This is the 1st statement and the other statements are jumbled up. Cut out the strips and re-construct the proof by arranging the remaining statements in the correct order. **(A)**
RED/BLUE and ODDS/EVENS GAMES Imagine we play the game with different objects (counters, cards, balls) of two types (two colours red and blue or numbered even and odd).

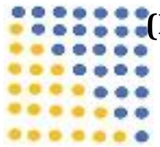
We have proved that: the game will be a fair game if and only if $(a - b)^2 = a + b$ **(B)**

- the total number of objects (i.e. $a + b$) has to be a square number, say n^2;
- the difference between the number objects of the two types (i.e. $a - b$) is n.

That is: $a + b = n^2$ (1)
and $a - b = n$ (2)

We have proved that a necessary and sufficient condition for a game of this sort to be a fair game is that the total number of objects ($a + b$) is a square number n^2, moreover a and b must be consecutive triangle numbers. **(C)**

This suggests that a and b are consecutive triangle numbers as the rule for generating the sequence of triangle numbers is that the difference between the n^{th} and $(n-1)^{th}$ triangle number is equal to n, and their sum is the square number n^2. **(D)**
(e.g. $T_7 - T_6 = 7$ and $T_7 + T_6 = 7^2$)



To Prove This is a sample space diagram (2-way table) It shows the total numbers of possible outcomes, grey for one outcome (e.g. different or odd) yellow for the other outcome (e.g. same or even). **(E)**

	a	b
a	$a^2 - a$	ab
b	ab	$b^2 - b$

	R_1	R_2	R_3	B_1	B_2	B_3	B_4	B_5	B_6
R_1	X	S	S	D	D	D	D	D	D
R_2	S	X	S	D	D	D	D	D	D
R_3	S	S	X	D	D	D	D	D	D
B_1	D	D	D	X	S	S	S	S	S
B_2	D	D	D	S	X	S	S	S	S
B_3	D	D	D	S	S	X	S	S	S
B_4	D	D	D	S	S	S	X	S	S
B_5	D	D	D	S	S	S	S	X	S
B_6	D	D	D	S	S	S	S	S	X

Notation: Label the number of objects of the different types as a and b with $a > b$. **(F)**
The symbol ⇔ means that the logical argument works both ways, the statement and its converse are both true, it's an 'if and only if' argument.

Adding the equations (1) and (2) : $2a = n^2 + n \iff a = ½(n^2 + n) \iff a = ½n(n+1)$ **(G)**
This formula gives a as the n^{th} triangular number.
Similarly, b is the $(n - 1)^{th}$ triangular number $½n(n-1)$.

If a and b are any two consecutive triangle numbers, the reverse argument holds.

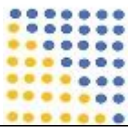
For a fair game we must have the probability of one outcome equal to the probability of the other outcome; that is: $(a^2 - a) + (b^2 - b) = ab + ab$ *(from the sample space diagram)* **(H)**

$\iff a^2 - 2ab + b^2 = a + b$ *(by rearranging this expression)*

$\iff (a - b)^2 = a + b$

Conclusion: Isomorphic games. The proof applies to all games that involve randomly picking two objects from a bag of objects of two different types where winning or losing corresponds to picking objects of the same or different types, for example, with red corresponding to odd and blue corresponding to even. The games are fair if and only if the number of objects of each type are consecutive triangle numbers a and b (so that $a + b = n^2$ and $a - b = n$). **(I)**

Conjecture: The game will be a fair game if and only if the numbers of objects of each type are consecutive triangle numbers and the total number of objects is a square number. **(J)**



INCLUSION AND HOME LEARNING GUIDE

THEME:

Probability Sample Spaces – Counting all possible outcomes

Lesson starter for a mixed-age group pre-school to 18 or 19 years

Game for Early Years: Pre School and Years 1 & 2

RED AND BLUE GAMES – any number of players

Resources: You need small objects of two different colours that are the same in all respects other than colour, such as counters or buttons or cards, and a bag, box, hat or envelope.

Put 3 red cards and 6 blue cards in a bag or envelope and take turns to pick 2 cards at random from the bag, one after the other without replacing the first one.

Each player must decide to be a 'same' player who wins when the cards drawn are the same colour or a 'different' player who wins when the two cards are different colours.

Pick 2 cards at random from the bag, score a point (or not) according to whether the cards are the same or different colours, and put the cards back in the bag.

Shake the bag before the next player takes a turn.

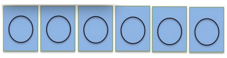
Repeat until one of the players has won 5 points.

Record the results as an experiment to discover if this is a fair game. After playing the game many times do your results seem to show that the players have an equal chance of drawing cards of the same colour as of drawing cards of different colours? Older learners can be given the task of explaining why they believe that there is an equal chance of drawing the same colour or different colours (or otherwise).

Lower Primary   Years 3 to 5

RED AND BLUE GAMES – any number of players

Resources: You need small objects of two different colours that are the same in all respects other than colour, such as counters or buttons or the cards cut out from page 5, and a bag or envelope.


Put 3 red cards and 6 blue cards in a bag or envelope and take turns to pick

2 cards at random from the bag.

Each player must decide to be a 'same' player who wins when the cards are the same colour or a 'different' player who wins when the two cards are different colours.

Pick 2 cards at random from the bag, give a point to the winner, according to whether the cards are the same or different colours, and put the cards back in the bag.

Shake the bag before the next player takes a turn.

Repeat until one of the players has won 5 points.

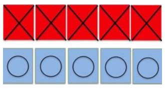
To discover if it is a fair game, note which team wins each time the class plays the game. Keep a record of the total number of times the cards chosen randomly are the same colour, and the number of times they are different colours. If it is a fair game then, the more often your class plays the game the the closer these two totals will get.

R_1	X	S	S	S	S	D	D	D	D	D
R_2	S	X	S	S	S	D	D	D	D	D
R_3	S	S	X	S	S	D	D	D	D	D
R_4	S	S	S	X	S	D	D	D	D	D
R_5	S	S	S	S	X	D	D	D	D	D
B_1	D	D	D	D	D	X	S	S	S	S
B_2	D	D	D	D	D	S	X	S	S	S
B_3	D	D	D	D	D	S	S	X	S	S
B_4	D	D	D	D	D	S	S	S	X	S
B_5	D	D	D	D	D	S	S	S	S	X

By Year 7 learners should be able to fill in a 2-way table like the one shown and to understand why it happens that players are more likely to draw two cards of different colours than two of the same colour.

Upper Primary   Years 6 and 7

IN YEAR 6 CHANGE THE GAME – PUT 5 RED CARDS AND 5 BLUE CARDS IN THE BAG.



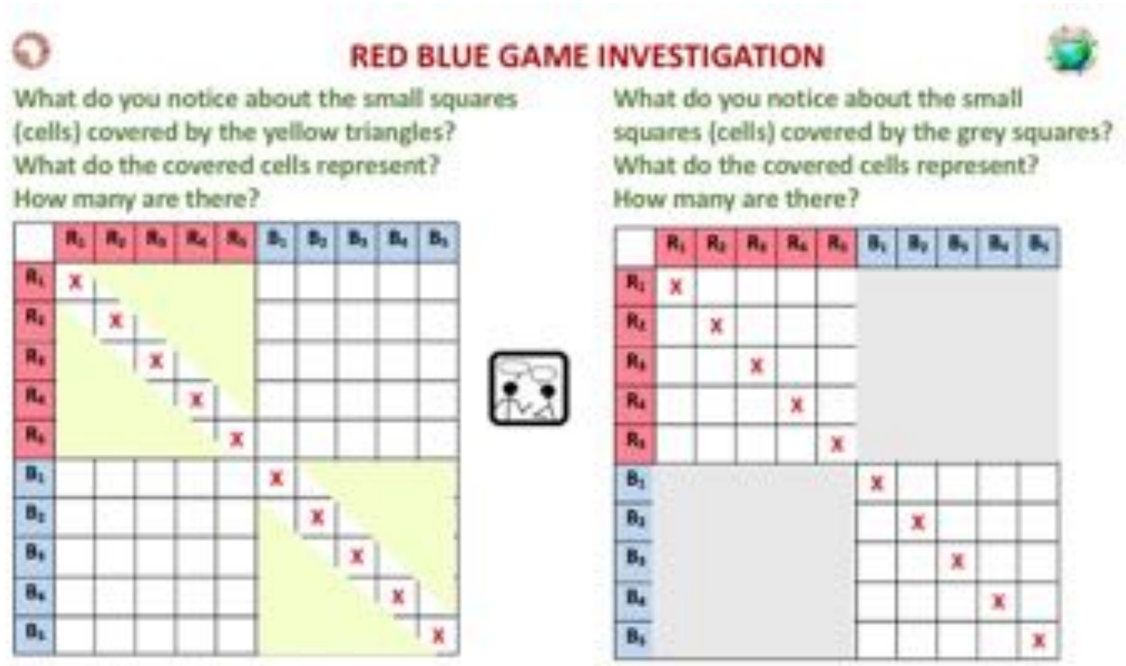
Play this game and investigate what happens. Record the results for each draw as 'same colour' or 'different colour'.

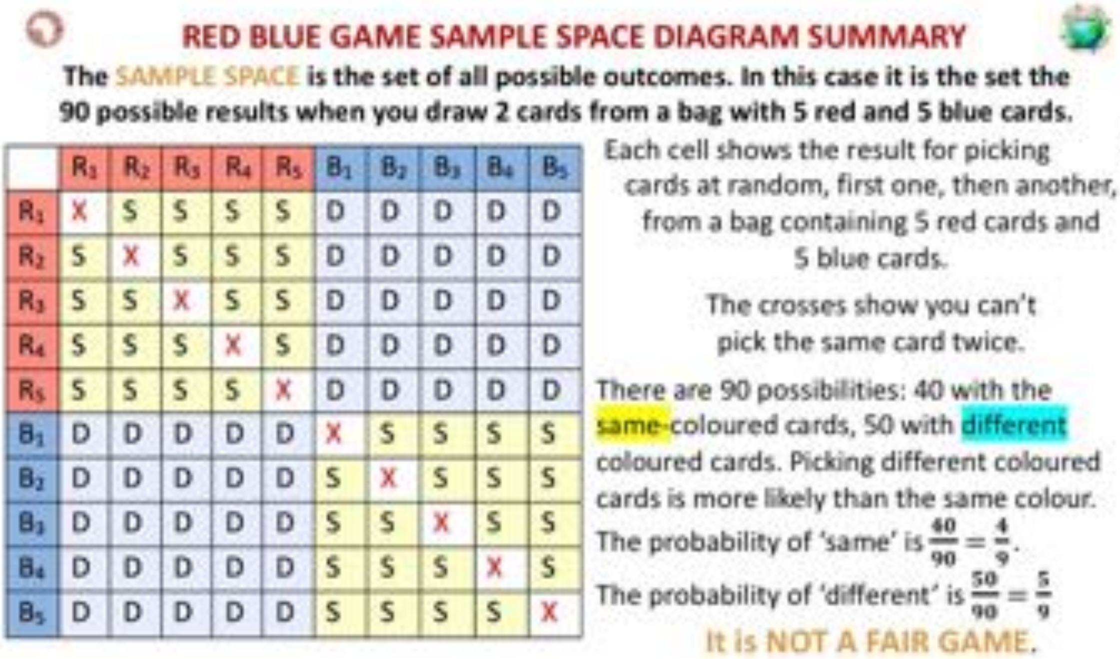
Many children will discover that this is NOT a fair game. They are more likely to get 2 cards of different colours than to get 2 cards of the same colour. If the teacher does not give away this 'secret winning strategy' then more and more members of the class will realize for themselves that they should choose to be 'different'. The teacher should encourage them to want to find out why.

To understand why players are more likely to draw two cards of different colours than two of the same colour, learners need to understand how to fill in a 2-way table showing all possible outcomes.

Ask the learners What do the red crosses on the diagonal line tell you?

	R_1	R_2	R_3	R_4	R_5	B_1	B_2	B_3	B_4	B_5
R_1	x									
R_2		x								
R_3			x							
R_4				x						
R_5					x					
B_1						x				
B_2							x			
B_3								x		
B_4									x	
B_5										x





RED BLUE GAME SAMPLE SPACE DIAGRAM SUMMARY

The SAMPLE SPACE is the set of all possible outcomes. In this case it is the set the 90 possible results when you draw 2 cards from a bag with 5 red and 5 blue cards.

	R_1	R_2	R_3	R_4	R_5	B_1	B_2	B_3	B_4	B_5
R_1	X	S	S	S	S	D	D	D	D	D
R_2	S	X	S	S	S	D	D	D	D	D
R_3	S	S	X	S	S	D	D	D	D	D
R_4	S	S	S	X	S	D	D	D	D	D
R_5	S	S	S	S	X	D	D	D	D	D
B_1	D	D	D	D	D	X	S	S	S	S
B_2	D	D	D	D	D	S	X	S	S	S
B_3	D	D	D	D	D	S	S	X	S	S
B_4	D	D	D	D	D	S	S	S	X	S
B_5	D	D	D	D	D	S	S	S	S	X

Each cell shows the result for picking 2 cards at random, first one, then another, from a bag containing 5 red cards and 5 blue cards.

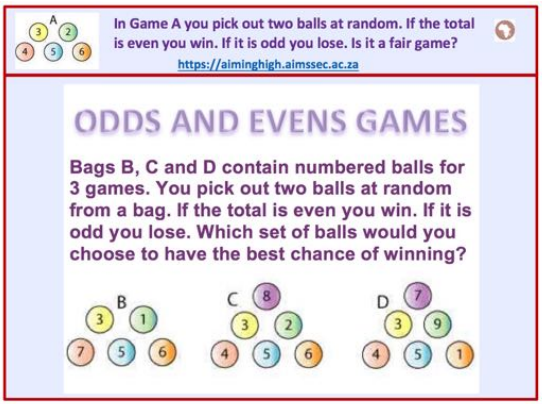
The crosses show you can't pick the same card twice.

There are 90 possibilities: 40 with the same-coloured cards, 50 with different coloured cards. Picking different coloured cards is more likely than the same colour.

The probability of 'same' is $\frac{40}{90} = \frac{4}{9}$.

The probability of 'different' is $\frac{50}{90} = \frac{5}{9}$

It is NOT A FAIR GAME.



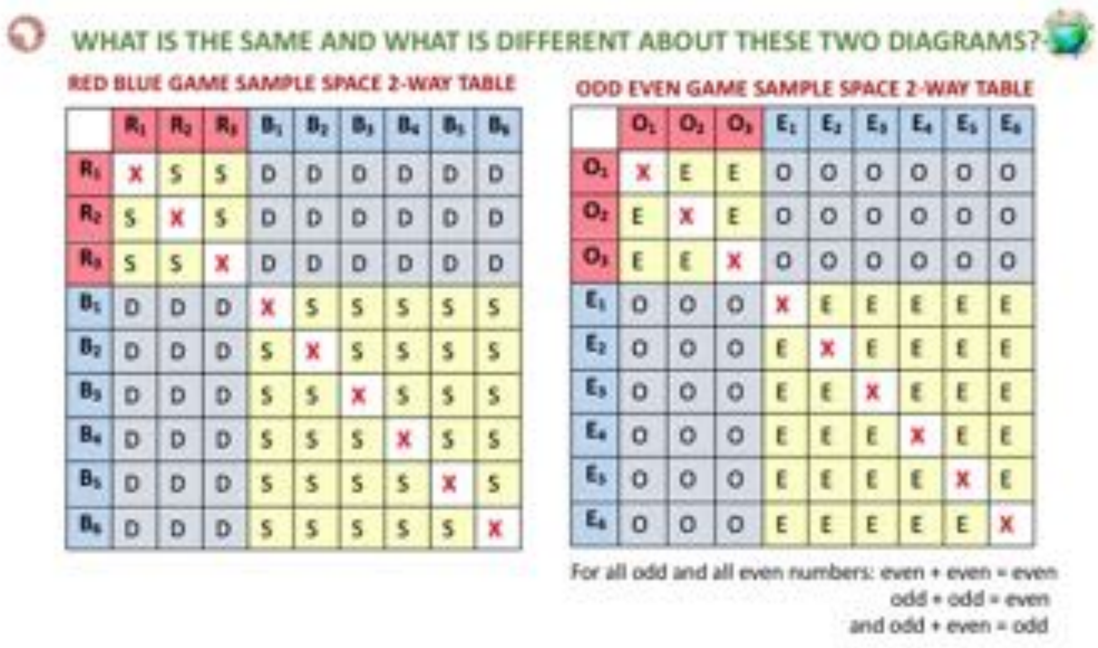
Introduce the Odds & Evens Games as on page 1 and in the worksheet. The 4 games are all played with the same rules.

Are any of the games fair?
Find a set of numbers that gives a fair game.
How do you know which sets of numbers give fair games?

Explore some of the properties of odd and even numbers. Tell the class to close their eyes and keep them closed until you tell them to open their eyes. Tell them:
ODD + ODD
Think of any two odd numbers. Add them. Is the sum odd or even?
Think of another two odd numbers. Add them. Is the sum odd or even?
Do this a few more times. Ask: What do you notice?
EVEN + EVEN
Think of any two even numbers. Add them. Is the sum odd or even?
Think of another two even numbers. Add them. Is the sum odd or even?
Do this a few more times. Ask: What do you notice?
ODD + EVEN
Think of any two numbers, one odd one even. Add them. Is the sum odd or even?
Think of another two numbers, one odd one even. Add them. Is the sum odd or even?
Do this a few more times. Ask: What do you notice?

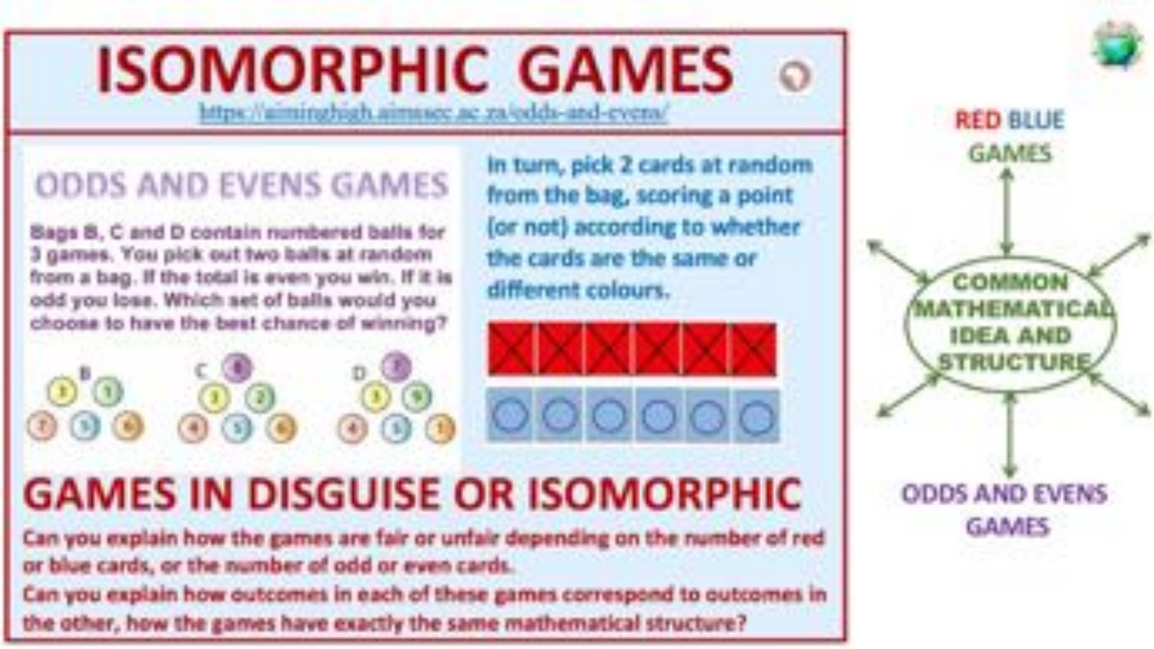
Lower Secondary for Years 7 – 10

Introduce both the Red Blue Games and the Odds and Evens Games then compare the different games and then explain the property of ISOMORPHISM between two mathematical systems.



	R_1	R_2	R_3	B_1	B_2	B_3	B_4	B_5	B_6
R_1	X	S	S	D	D	D	D	D	D
R_2	S	X	S	D	D	D	D	D	D
R_3	S	S	X	D	D	D	D	D	D
B_1	D	D	D	X	S	S	S	S	S
B_2	D	D	D	S	X	S	S	S	S
B_3	D	D	D	S	S	X	S	S	S
B_4	D	D	D	S	S	S	X	S	S
B_5	D	D	D	S	S	S	S	X	S
B_6	D	D	D	S	S	S	S	S	X

	O_1	O_2	O_3	E_1	E_2	E_3	E_4	E_5	E_6
O_1	X	E	E	O	O	O	O	O	O
O_2	E	X	E	O	O	O	O	O	O
O_3	E	E	X	O	O	O	O	O	O
E_1	O	O	O	X	E	E	E	E	E
E_2	O	O	O	E	X	E	E	E	E
E_3	O	O	O	E	E	X	E	E	E
E_4	O	O	O	E	E	E	X	E	E
E_5	O	O	O	E	E	E	E	X	E
E_6	O	O	O	E	E	E	E	E	X



Suggestions for teaching Odds and Evens (see page 1)

Resources *Each pair should have a copy of page 1. You can simulate the games using numbered counters or numbered cards (see page 2) in a bag, or spinners (see page 3).*

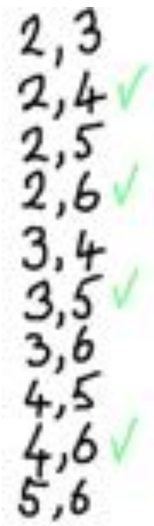
Start with the diagnostic question so that everyone will then have sample space diagrams (2-way tables) in mind when they tackle the activity.

The notes that follow are based on lessons with a group of 14 year old students. The notes are in two parts: the first part for those who wish to use the activity for a single lesson on probability and sample space diagrams or tree diagrams. The second part is for those who wish to follow this up with a collaborative task that leads to interesting and unexpected results, and this is also suitable for older students.

LESSON 1 Start by playing Game A using numbered cards in a bag or using the computer simulation http://nrich.maths.org/4308. The computer simulation generates lots of experimental data quickly, freeing time to focus on predictions, analysis and justifications. Calculating the theoretical probabilities provides a motivation for using sample space diagrams or perhaps tree diagrams.

Play the game no more than ten times, so that everyone gets 'a feel' for the game but doesn't have sufficient results to draw conclusions about the probabilities. Then ask everyone to decide whether they think the game is fair, and ask them to do some maths to support their decision.

While everyone is working, the group leader or teacher should circulate and observe the methods of recording results being used, such as:



At this point the younger children could play the RED AND BLUE GAMES and the young people of 16 or older could work on the tasks as explained in the section for UPPER SECONDARY.

The group should then discuss their methods of trying to find out whether the ODDS & EVENS GAME A is a fair game. Choose individuals who used different methods to explain to everyone else what they did, recording what they did on the board. Compare the different methods and emphasise the merits of a 2-way table, sample space method rather than a listing method. Those who are confident with tree diagrams may prefer to continue using them.

If you have the internet, use the computer interactivity a large number of times to confirm that the experimental probability matches closely to the theoretical probability that has been calculated. There are opportunities here for rich discussion about how closely we expect an experimental probability to match the theory.

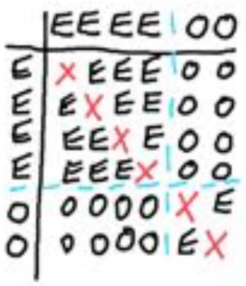
HOMEWORK OR FOLLOW-ON QUESTION FOR THE NEXT LESSON:

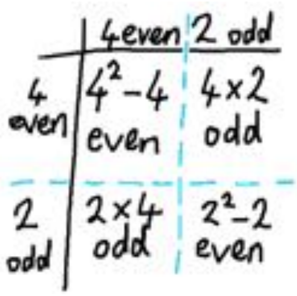
Now work on GAMES B, C and D, and everyone should **think on their own, without writing,** about which of the four games they would choose to play, to maximise their chances of winning and work out the respective probabilities. Next lesson, once everyone has had a short time to reflect, **work in pairs** to discuss the choices people have made, and to justify decisions (again, without writing). There is often disagreement about which game offers the best chance of winning, so **the group together should compare ideas** and try to use efficient listing methods.

Lesson 2: Once the probabilities have been calculated for Games, A, B, C and D, you might use the interactivity http://nrich.maths.org/4308 again to do some experimental trials. To confirm that the experimental probability is close to the calculated one, you would have to do thousands of random trials.

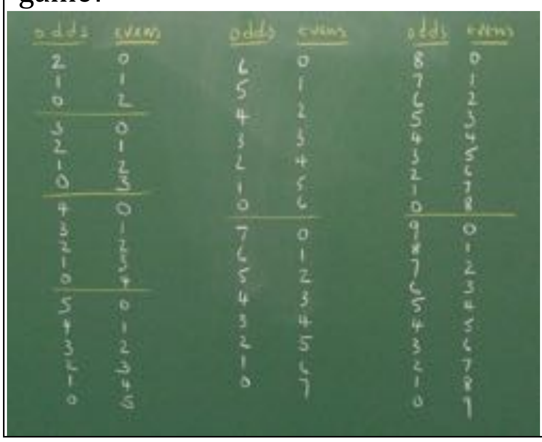
Now write up on the board a set E, which contains four large even numbers and two large odd numbers. Make them large enough that calculations would be off-putting! Ask the group to work in pairs to calculate the probability of winning with set E. The intention is to alert everyone that the numbers themselves don't matter, but the numbers of odds and evens is the important point. Set E has the same structure as Set C, so we already know the chance of winning.

Then the group can be introduced to this sort of sample space diagram where odds and evens are collected together, and discuss and explain the entries in the tables:





None of the sets looked at so far (A, B, C, D and E) gives a fair game.
Ask "How could we find out whether there are any sets of numbers that give a fair game?"



One way of organising the search is to draw up a table on the board showing different combinations of odds and evens.

Those already identified as not being fair games (sets A, B, C and D) can be crossed off.

Then split into pairs to work on different combinations and report back to the whole group

Record combinations that have been checked on the board with a tick or a cross to show whether they are fair or not. If something has two ticks or two crosses, it could be accepted as being confirmed.

When disagreements arise, ask other pairs to resolve them.

There will be opportunities while everyone is working to stop and share insights that will make the job easier.

For example:

"None of the combinations with zero will work because..."

"If 3 odds and 2 evens won't work, 2 odds and 3 evens won't either, because..."

"You can't have the same number of evens and odds because..."

Eventually, there will be a sea of crosses on the board and just a few combinations that work (four, if the group has gone up to 9 balls in total). Stop and consider what the fair sets have in common. This may lead to some new conjectures about the total number of balls, so test the conjecture out on the next obvious total.

Once there is some confirmation about the total number of balls needed for fair games, conjectures can also be made about how these should be split into odds and evens. People can set to work to test examples with large numbers, using the simplified sample space method above. Talk about how valuable it is to work collaboratively as part of a mathematical community, and how difficult it would have been to have reached the same insights working alone.

Key questions

- How can you decide if a game is fair?
- What are the most efficient methods for recording possible combinations?
- How can we make this difficult task (of finding a fair game) more manageable?

Upper Secondary

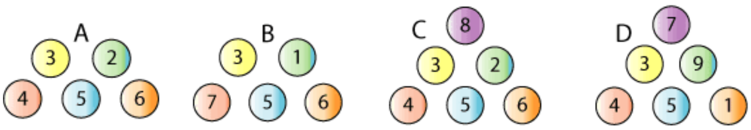
Many 'real world' situations (such as clinical trials in medical research) cannot be analysed theoretically and have to depend on experimental probability, with a large number of trials on subjects drawn randomly from the population, so as to provide a mathematical model for the real situation.

In doing this activity you will explore two types of probability: experimental and theoretical by playing 4 games A, B, C and D, trying to discover by experimenting whether they are fair games, and then analysing the games to work out the theoretical probabilities of drawing numbers adding to odd and even totals.

After that you will meet isomorphism (mathematical systems that look different but are really the same) and you will apply some simple algebra to prove a surprising result.

Work through the activities on page 1, if possible with a partner or group. Start by playing Game A. Use numbered cards in a bag or use the computer simulation

http://nrich.maths.org/4308. The computer simulation generates lots of



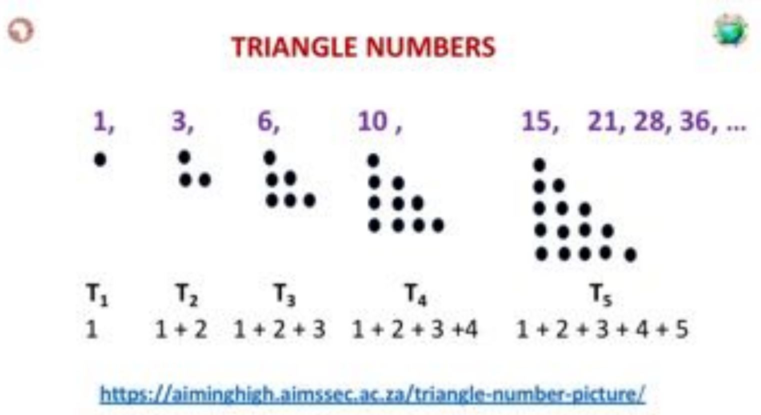
experimental data quickly, freeing time to focus on predictions, analysis and justifications. To calculate the theoretical probabilities use sample space diagrams like the one in the HELP box.

Work on Games B, C and D and prove to yourself that none of these games are fair games because, in every case, the probability of getting an odd score is not equal to the probability of getting an even score.

Create sets of numbers for which the Odds and Evens game would be a fair game.

Formulate conjectures about what conditions lead to a fair game and try to prove your conjectures.

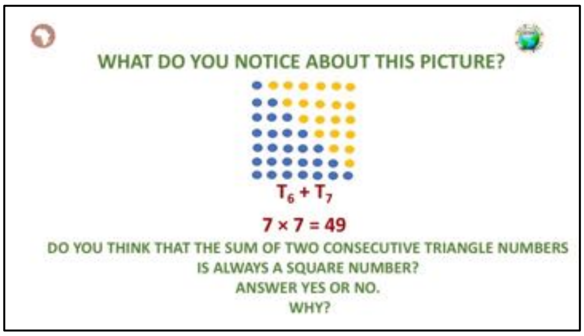
The very simplest arithmetic series

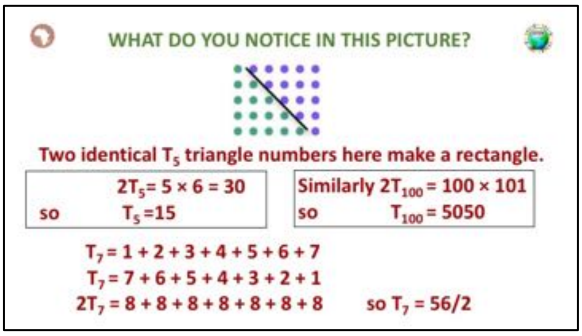


Numbers	1	2	3	4	5	...	99	100
Reversed	100	99	98	97	96	...	2	1
Totals	**101**	**101**	**101**	**101**	**101**		**101**	**101**

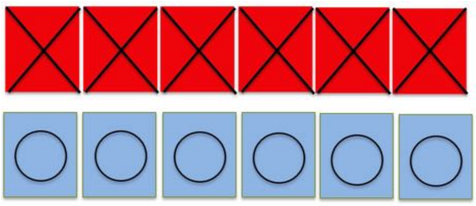
What do you notice about this table? Does it tell you how to prove the formula for the sum of n terms of the arithmetic series 1 + 2 + 3 + 4 +... ?

Look closely, what do you notice about the diagrams of triangle numbers in the two illustrations below?





Next cut out the strips on page 5. Arrange them in the correct order to re-create the proof of the conditions for the numbers of odd and even cards to lead to a fair game.



Play the Red and Blue Games (see page7). Explain how these games are isomorphic to the Odd and Even Games. 'Isomorphic' means that the two systems have exactly the same structure. The only difference is the context of 2 colours mimicking the odd and even numbers. The word is derived from the Greek 'iso' meaning the same and 'morph' meaning change.

Key questions

- How can you decide if a game is fair?
- What are the most efficient methods for recording possible combinations?
- How can we make this difficult task (of finding a fair game) more manageable?

Now is the time for abstract thinking and relying on the power of algebra

Proof Sorting Exercise for Year 12 and 13 Students

RED/BLUE and ODDS/EVENS GAMES Imagine we play the game with different objects (counters, cards, balls) of two types (two colours or numbered even and odd.

Can you prove the conjecture: The game will be a fair game if and only if the numbers of objects of each type are consecutive triangle numbers and the total number of objects is a square number.

Cut out the strips on page 4 and re-construct the proof by arranging the statements in the correct order.

Why do this activity?

In this activity learners can explore and discuss two types of probability: experimental and theoretical. They can experience teamwork in trying to find an example of a fair game of this type and trying to formulate conjectures about what conditions might lead to a fair game. Older learners can work on the challenge of proving the conjectures.

Learning objectives

In doing this activity students will have an opportunity to:

- conduct probability experiments and record results;
- experience and discuss random sampling;
- learn to use a sample space diagram to show all the elements in a sample space and to analyse the theoretical probabilities of different events;
- reflect on the differences between experimental probability and theoretical probability;
- find and prove conjectures about conditions for a fair game.

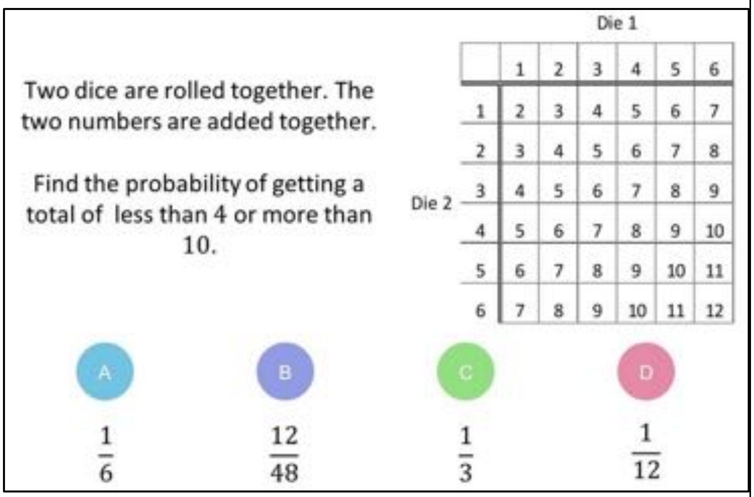
Generic competences

In doing this activity students will have an opportunity to:

- conduct probability experiments, reflect on the differences between experimental probability and theoretical probability and discuss the types of applications that require the use of one or the other;
- work as a team towards a common goal;
- **communicate** in writing, speaking and listening:
    - exchange ideas, criticize, and present information and ideas to others;
    - analyze, reason and record ideas effectively.

Diagnostic Assessment This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 for C and 4 for D".
2. Notice how the learners respond. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. It is important for learners to explain the reason for their answer to clarify their own thinking and to practise communication skills.
4. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
5. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.**


Two dice are rolled together. The two numbers are added together.

Find the probability of getting a total of less than 4 or more than 10.

Die 1

Die 2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

A $\frac{1}{6}$ B $\frac{12}{48}$ C $\frac{1}{3}$ D $\frac{1}{12}$

6. The concept is needed for the lesson to follow, so explain the right answer or give a remedial task.

The correct answer is **A.** There are 6 possible outcomes out of 36.

Common Misconceptions: B. May be a guess

C. These learners might have included scores of 4 & 10 or misunderstood 'less than'.

D. May be a guess https://diagnosticquestions.com

SOLUTION

Game A

Probability of an even sum is $\frac{8}{20} = \frac{2}{5}$.
Probability of an odd sum is $\frac{12}{20} = \frac{3}{5}$.
So it is not a fair game.

	2	4	6	3	5
2	X	6	8	5	7
4	6	X	10	7	9
6	8	10	X	9	11
3	5	7	9	X	8
5	7	9	11	8	X

Game B

Probability of an even sum is $\frac{12}{20} = \frac{3}{5}$.
Probability of an odd sum is $\frac{8}{20} = \frac{2}{5}$.
So it is not a fair game.

	1	3	5	7	6
1	X	4	6	8	7
3	4	X	8	10	9
5	6	8	X	12	11
7	8	10	12	X	13
6	7	9	11	13	X

Game C

Probability of an even sum is $\frac{14}{30} = \frac{7}{15}$.
Probability of an odd sum is $\frac{16}{30} = \frac{8}{15}$.
So it is not a fair game.

	2	4	6	8	3	5
2	X	6	8	10	5	7
4	6	X	10	12	7	9
6	8	10	X	14	9	11
8	10	12	14	X	11	13
3	5	7	9	11	X	8
5	7	9	11	13	8	X

Game D

Probability of an even sum is $\frac{21}{30} = \frac{7}{10}$.
Probability of an odd sum is $\frac{9}{30} = \frac{3}{10}$.
So it is not a fair game.

	1	3	5	7	9	4
1	X	4	6	8	10	5
3	4	X	8	10	12	7
5	6	8	X	12	14	9
7	8	10	12	X	16	11
9	10	12	13	16	X	13
4	5	7	9	11	13	X

Odds and Evens Games Theorem and Proof

RED/BLUE and ODDS/EVENS GAMES Imagine we play the game with different objects (counters, cards, balls) of two types (two colours red and blue or numbered even and odd).

Notation: Label the number of objects of the different types as a and b with $a > b$.

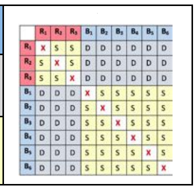
The symbol $\Leftrightarrow$ means that the logical argument works both ways, the statement and its converse are both true, it's an 'if and only if' argument.

Conjecture: The game will be a fair game if and only if the numbers of objects of each type are consecutive triangle numbers and the total number of objects is a square number.

To Prove:

This is the sample space diagram (2-way table). It shows the total numbers of possible outcomes, with grey for one outcome (e.g. different or odd) and yellow for the other (e.g. same or even).

	a	b
a	$a^2 - a$	ab
b	ab	$b^2 - b$



For a fair game we must have the probability of one outcome equal to the probability of the other outcome; that is: $(a^2 - a) + (b^2 - b) = ab + ab$ *(from the sample space diagram)*

$\Leftrightarrow \quad a^2 - 2ab + b^2 = a + b$ *(by rearranging this expression)*

$\Leftrightarrow \quad (a - b)^2 = a + b$

We have proved that: the game will be a fair game if and only if $(a - b)^2 = a + b$

- the total number of objects (i.e. $a + b$) has to be a square number, say n^2;
- the difference between the number objects of the two types (i.e. $a - b$) is n that is $a + b = n^2$ (1) and $a - b = n$ (2)

This suggests that a and b are consecutive triangle numbers because the rule for generating the sequence of triangle numbers is that the difference between the n^{th} and $(n - 1)^{th}$ triangle number is equal to n, and their sum is a square number.

Adding the two equations: $2a = n^2 + n \Leftrightarrow a = ½ (n^2 + n) \Leftrightarrow a = ½n (n+1)$
This formula gives a as the n^{th} triangular number.
Similarly, b is the $(n - 1)^{th}$ triangular number $½n (n -1)$.

If a and b are any two consecutive triangle numbers, the reverse argument holds.

We have proved that a necessary and sufficient condition for a game of this sort to be a fair game is that the total number of objects $(a + b)$ is a square number n^2, moreover a and b must be consecutive triangle numbers.

Conclusion: Isomorphic Games. The proof applies to all games that involve randomly picking two objects from a bag of objects of two different types where winning or losing corresponds to picking objects of the same or different types, for example: same or even red versus different or odd. The games are fair if and only if the number of objects of each type are consecutive triangle numbers a and b (so that $a + b = n^2$ and $a - b = n$).

Follow up

The activity In a Box https://aiminghigh.aimssec.ac.za/in-a-box/ offers another context for exploring exactly the same game and underlying mathematical structure. Use it as a follow-up a few weeks after working on Odds & Evens.

Also see Special Sums https://aiminghigh.aimssec.ac.za/special-sums/

Red or Black: https://aiminghigh.aimssec.ac.za/red-or-black-game/

Nines or Tens https://aiminghigh.aimssec.ac.za/nines-and-tens/

Two Aces https://aiminghigh.aimssec.ac.za/two-aces/

In the Bag https://aiminghigh.aimssec.ac.za/in-the-bag/

Twos Company https://aiminghigh.aimssec.ac.za/twos-company/

Same Sweets https://aiminghigh.aimssec.ac.za/same-sweets

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and school years up to Secondary 5 in East Africa.
New material will be added for Secondary 6.
For resources for teaching A level mathematics (Years 12 and 13) see https://nrich.maths.org/12339
Mathematics taught in Year 13 (UK) & Secondary 6 (East Africa) is beyond the SA CAPS curriculum for Grade 12

	Lower Primary Approx. Age 5 to 8	Upper Primary Age 8 to 11	Lower Secondary Age 11 to 15	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13