## AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES <br> SCHOOLS ENRICHMENT CENTRE (AIMSSEC)

AIMING HIGH
This INCLUSION AND HOME LEARNING GUIDE suggests related learning activities for all ages from 4 to 18 on the theme of SCALE

## Choose what seems suitable for the age or attainment level of your learners

 The original SIERPINSKI NUMBER \& SHAPE PATTERNS was designed for Years 6 to 12
## SIERPISKI NUMBER AND SHAPE PATTERNS



The first question is how many small tetrahedra, like the blue model shown, were used to make this 6.5 metre high red balloon model tetrahedron. It was built in a shopping mall in Cambridge, UK, to set a Guinness World Record.

The smallest tetrahedron, the blue model (Stage 0 ) is made from 6 balloons, each 25 centimetres long.


The green tetrahedron

(Stage $1-50 \mathrm{~cm}$ ) made from four 25 cm pyramids, would have edges of length 50 centimetres in a perfect model.

Four of the Stage 1 tetrahedra are used to make the Stage 2 ( 1 metre) model, and four of those to make the next one, Stage 3 ( 2 metres) and so on, and so on. At each stage of the construction four tetrahedra are used to make a bigger tetrahedron and the lengths of the edges are double the lengths of the edges at the previous stage.

This white 3D printed model shows the construction of the ideal red balloon model.

The red balloon model was made from 1024 small tetrahedra with edges of length 25 centimetres. Clearly it was not a perfect shape like the white model, but if it had been then the edges would have measured 8 metres.

Note on Language: Tetrahedron is derived from the Greek tetra (four) and hedron (face). Similarly octahedron is derived from octa (eight) and polyhedron from poly (many).


The rainbow tetrahedron is a Stage 4 (4 metre) construction made with 4 of the Stage 3 ( 2 metre) tetrahedra.
There are many other questions about scale, number patterns and geometry that you can investigate based on this structure.

For example: 'How many triangular faces are there at each stage?' 'If the (Stage 0) tetrahedra, like the violet tetrahedra on the bottom layer, were solid rather than skeletal, what fraction of the volume of the tetrahedron would be filled at each stage and what fraction would be empty space?'
'What shape is the hole in the middle?'
Let us know what you find out.

## HELP SIERPINSKI TRIANGLE (The same construction in 2 dimensions)

Use triangular grid-paper if possible and make your fractal as big as possible.

Step One Draw an equilateral triangle with sides of 2 triangle lengths each. Connect the midpoints of each side. How many
 equilateral triangles do you now have?
Shade the three outer triangles. Alternatively, think of this as cutting a triangular hole in the triangle.

You could imagine that all the components in the diagrams
 below are scaled down to reduce them in size or, alternatively, that more and more triangular holes are made at each stage.

Step Two Put 3 of these patterned triangles together. Alternatively draw another equilateral triangle with sides of 4 triangle lengths each and mark the edges into 4 equal lengths. Shade the nine small triangles as shown. Note the larger 'hole' and the three smaller 'holes'.

Step Three Put three of these patterned triangles together, that is nine of the patterned triangles from Step One. Alternatively draw an equilateral triangle with sides of 8 triangle lengths. Shade the 27 small triangles as shown. You will have 1 large, 3 medium, and 9 small 'holes'. Note the number pattern 1, 3,
 9, 27,...

Step Four Make a poster? Each member of the group should make one of the patterned triangles from Step Three, then 3 triangles can be assembled to make a patterned triangle as in the diagram below, or 9 or 27 (or even 81...) can be assembled to make patterns getting closer to a Sierpinski
 Triangle Fractal.
Use your artistic creativity to shade the triangles in interesting colour patterns.

## NEXT

To develop your investigative and communication skills, choose and investigate one aspect of the construction of a giant Sierpinski Tetrahedron and write up your findings. Different learners could pursue different investigations to answer different questions (see the list in the 'Why do this activity' section). A project like this offers the opportunity for you to prepare a report, to make your own models and posters and give a talk to present your discoveries to the whole class.

One choice of project could be to find out more about Waclaw Sierpinski (1882-1969) and his work. He was born in Warsaw, Poland, when it was part of the Russian Empire. He was a mathematician, specializing in set theory, topology and number theory, who experienced the hardships of the two World Wars. He wrote 724 research papers and 50 books. Warsaw University Library and Sierpinski's own house, including his library and personal papers, were burnt down by the Nazis and many of his mathematician colleagues and friends were killed. He retired in 1960 at the age of 78 but continued with his editorial work and lecturing.



A fractal is a shape or curve that contains within itself infinitely many copies at smaller and smaller scales.

This Illustration shows 4 stages of Sierpinski's space filling curve. In the limit the fractal curve passes through every interior point of the square. However, the area enclosed by the curve is less than half the area of the square.

We would need to deepen our understanding of length and area, through further study, to fully appreciate ideas like space filling curves. Length and area are not the simple concepts that they appear to be in school mathematics.

## INCLUSION AND HOME LEARNING GUIDE THEME: SCALE

## Early Years and Lower Primary - How many black triangles?

How many black triangles are there in this pattern?


What do you notice about the black triangles in this pattern? How many black triangles of each size are there?


What do you notice about the black triangles in this pattern? In what way is it the same and how is it different from the patterns in the first two triangles above?
How many black triangles of each size are there?


What do you notice about the black triangles in this pattern? How many black triangles of each size are there?
What patterns can you spot/


How many white triangles are there in these diagrams. How many triangles are there altogether.

## Upper Primary - Sierpinski Triangle Poster

You will need a large sheet of backing paper fixed to the wall and a gluestick.


- Give each learner a copy of this Sierpinski triangle and ask them to colour it according to this pattern and
 to cut it out.
- Draw a very faint line across your backing paper near bottom edge.

Ask learners to come one by one and glue the back of the triangle that they have coloured. Then show them where to glue their triangle in the poster. Glue 2 triangles on the line you have drawn near the lefthand edge of your backing sheet. Then glue another triangle on top to make this pattern.


Your poster now shows a Stage 4 Sierpinski Triangle.
Use 3 more learner's triangles to form another Stage 4 triangle on the bottom line touching the first one, then another three triangles on top to make a Stage 5 triangle, using altogether 9 triangles like the orange diagram.

If your backing paper is large enough you can use 27 triangles to make a Stage 6 Sierpinski triangle or even 81 triangles to make a Stage 7.


## Lower Secondary

Teachers and others who guide children's learning must be selective because there is a wealth of mathematical content here related to the Sierpinski Tetrahedron for all ages. They must decide on the learning objective for their lesson and focus the lesson accordingly.

There is considerable advantage in coming back to this same Sierpinski Model on different occasions to pursue different learning objectives in different lessons.

Your learning objective might be the development of investigative and communication skills. Then this activity could be the basis of mathematical project work. Different learners could pursue different investigations to answer different questions (see the list in the 'Why do...' section). A project like this offers the opportunity for learners to prepare reports, models, posters and talks to present to the whole class.

You might like to start with the Sierpinski Tetrahedron (below) or the Sierpinski Triangle activity (See HELP section).


Learners could make their own model Sierpinski tetrahedron in order to understand the repetitive process of construction and to help them to visualize the patterns.


Make four tetrahedra. Fix three tetrahedra together on a flat base to make a larger triangular base. Stick the fourth tetrahedron on top.
It is not necessary to go any further than this first stage of the process but the class could make a bigger model by putting together models made by different learners.
There are many materials you could use: rolled paper sticks tied together by string; tetrahedra made from folding paper, toothpicks fixed together using plasticine or Prestik or Blu Tack, cotton bud tubes threaded together, drinking straws and pipe cleaners, or balloons.

## QUESTIONS FOR MATHS LESSONS

1. Look at the second triangle in Step 1 in the HELP section. What fraction of the triangle is shaded? What fraction is unshaded?
2. You can think of the diagrams of the Sierpinski Triangle on page 2 in two different ways. As drawn they show a process of making more triangular holes at smaller and smaller scales, in stages that, carried on forever would produce the Sierpinski fractal. Alternatively these diagrams of the Sierpinski Triangle could have been reduced in scale to fit more of models that were growing in size, into the same actual space. What are the linear scale factors in these reductions? What are the area scale factors?
3. What fractions of the triangle are shaded in Steps 2,3 and 4 ? What fractions are unshaded?
4. Do you see a pattern here? Use the pattern to predict the fraction of the triangle you would shade in a poster made with 9 patterned triangles from Step 3?
5. Confirm your prediction and explain it.
6. What fraction is shaded in a poster made with 27 of the patterned triangles from Step 3?
7. Write the fractions of the triangles shaded in the above steps in order from least to greatest. Write a statement about how their order connects to the shading?
8. CHALLENGE 2D: Develop a rule or formula so that you could calculate the fraction of the area which is shaded for any step.
9. Find another interesting pattern in the Sierpinski Triangle. Write a paragraph describing your pattern.
10. CHALLENGE 3D: If the Scale 0 tetrahedron were solid (rather than skeletal) how many times greater would be the volume of the record breaking tetrahedron compared to the volume of the original small tetrahedron? Answer: the volume scale factor is $\left(2^{5}\right)^{3}=32768$
11. There is a big hollow space inside each Sierpinski Tetrahedron. What is the shape of that space? It is certainly not another tetrahedron. Have a look and see for yourself.

## Key questions

- What do you see?
- Can you describe the shapes you see in the model?
- Can you describe how the model is made?
- Can you find similar shapes?
- How many edges?
- How many tetrahedra of each size at each stage?
- What are the edge lengths at each stage?


## Upper Secondary

Do the activities on pages 1 and 2 . Then, as suggested in the NEXT section on page 3, choose your own question to investigate relating to the Sierpinski Tetrahedron and prepare a poster or presentation that you can do for your group or your class in school, or for a group of younger students. Alternatively, you could publish your presentation in a blog.

## Why do these activities?

They offer motivation and scope for mathematical work at all levels including University level:

1. Language development - all ages. What do you see? Can you describe how the model is made?
2. Investigate number patterns - all ages. For example, some of the simplest patterns are: $1,2,4,8,16, \ldots$ also $1,4,16,64, \ldots .1,3,9,27, \ldots$ and $6,24,96, \ldots$
3. Ideas of infinite processes and limits. "What if this process continued on and on getting bigger and bigger?' "Can you imagine and describe this process continued on and on getting smaller and smaller?' What other theoretically infinite processes in mathematics do you know about?
4. For the older students: find and sum geometric series, both with terms getting larger and also with terms getting smaller.
5. Study the Sierpinski models as an introduction to fractals and links with other fractal structures and fractal-like structures in nature.
6. Describe the properties of equilateral triangles and regular tetrahedra.
7. Investigate the 3D geometry in the Sierpinski Tetrahedron, and measures and calculation of lengths and areas with applications of Pythagoras Theorem.
8. Investigate similarity, enlargement and scale factors. For example: If the small Stage 0 tetrahedra are solid (rather than skeletal) how many times greater would be the volume of the record breaking tetrahedron be compared to the volume of the original small tetrahedron? Answer: the volume scale factor is $\left(2^{3}\right)^{5}=262144$ can you explain that?
9. There is a big hollow space inside each Sierpinski Tetrahedron. What is the shape of that space? It is certainly not another tetrahedron. Have a look and see for yourself.
10. Use a spreadsheet for the calculations.

Watch and enjoy this Mandelbrot Zoom video:
https://www.youtube.com/watch?v=0jGaio87u3A
It all depends on a simple equation, but to understand the mathematics behind it you'll have to study complex analysis and iteration theory at a higher level well beyond school mathematics.

## SOLUTIONS

Here are some answers to help teachers to use this activity as a starting point for many different mathematical activities and investigations. There is scope at all levels from developing language, scope for young learners just describing what they see, and for upper secondary level to work on calculations involving scaling, trigonometry, geometric series and ideas of limits and infinity.


The photo shows the Guinness World Record Breaking Balloon
Model constructed in a shopping mall in Cambridge UK in March 2014 as a fundraiser for AIMSSEC, by a team led by Bubblz the Mathematical Clown.

Clearly it was not a perfect shape


Regular Tetrahedron


One Face of Regular Tetrahedron

| STAGE | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Number of tetrahedra | 1 | 4 | $4^{2}=16$ | $4^{3}=64$ | $4^{4}=256$ | $4^{5}=1024$ | $4^{6}=4096$ |
| Number of 25 cm balloons | 6 | 24 | 96 | 384 | 1536 | 6144 | 24576 |
|  |  |  |  |  |  |  | Target for new record |
| Edge length a metres | 0.25 | 0.5 | 1 | 2 | 4 | 8 | 16 |
| Altitude of triangula face $\mathrm{L}=\mathrm{a} \sqrt{3} / 2$ metres | 0.217 | 0.433 | 0.866 | 1.732 | 3.464 | 6.928 | 13.856 |
| $\begin{gathered} \text { Verticalheight } \\ H=a \sqrt{ }(2 / 3) \text { metres } \end{gathered}$ by Pythagoras Th. | 0.204 | 0.408 | 0.816 | 1.633 | 3.266 | 6.532 | 13.064 |
| Vertical height in feet | 0.670 | 1.339 | 2.679 | 5.358 | 10.715 | 21.430 | 42.460 |

## Key Questions

- What do you see?
- Can you describe the shapes you see in the model?
- Can you describe how the model is made?
- Can you find similar shapes?
- How many edges are there at each stage?
- How many tetrahedra of each size are there at each stage?
- What are the edge lengths at each stage?
- Ask as many other questions as you can think of about the Sierpinski Tetrahedron.


## Why do this activity?

This activity offers motivation and scope for mathematical work at all levels:

1. Language development - all ages. Ask 'What do you see? Can you describe how the model is made?
2. Number patterns - all ages. $1,2,4,8,16, \ldots$ also $1,4,16,64, \ldots .1,3,9,27, \ldots$ and $6,24,96, \ldots$
3. Ideas of infinite processes and limits. Ask "What if this process continued on and on getting bigger and bigger?' "Can you imagine this process continued on and on getting smaller and smaller?'
4. For the older students: summing geometric series, both with terms getting larger and also with terms getting smaller.
5. This leads to understanding fractals and links with other fractal structures and fractals in nature.
6. Recognition and properties of equilateral triangles and regular tetrahedra.
7. 3D geometry, measures and calculation of lengths and areas. Applications of Pythagoras Theorem.
8. Similarity, enlargement and scale factors. For example: If the small Stage 0 tetrahedra are solid (rather than skeletal) how many times greater would be the volume of the record breaking tetrahedron be compared to the volume of the original small tetrahedron? Answer: the volume scale factor is $\left(2^{3}\right)^{5}=262144$
9. There is a big hollow space inside each Sierpinski Tetrahedron. What is the shape of that space? It is certainly not another tetrahedron. Have a look and see for yourself.
10. Use of a spreadsheet for the calculations.

## Learning objectives

In doing this activity students will have an opportunity to:

- investigate Number Patterns, for example: $1,2,4,8, \ldots \ldots$ and $6,24,96$, also $1,4,16, \ldots$ and $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ and $1,3,9,27, \ldots$ and $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ (all ages);
- work on summing geometric series (for the older students);
- gain some understanding of infinite processes and limits;
- develop the visualization skills to recognize equilateral triangles and regular tetrahedra when viewed from different angles and when 3D structures are represented in 2D;
- develop understanding of properties of 2D and 3D shapes;
- work on 3D geometry, measures and calculation of lengths, areas and volumes including applications of Pythagoras Theorem;
- gain better understanding of similarity, enlargement and scale factors.


## Generic competences

In doing this activity students will have an opportunity to:

- develop language and communication skills;
- develop geometric thinking and the ability to reason from pictures and diagrams;
- gain some understanding of fractals and the links with other fractal structures and fractals in nature;
- gain better understanding of similarity, enlargement and scale factors as used in map scales and design drawings;
- develop skills in using Excel spreadsheets for calculations.

DIAGNOSTIC ASSESSMENT This should take about 5-10 minutes.
Choose the Quiz most suitable for your class, write the question on the board, say to the class:
"Put up 1 finger if you think the answer is $A, 2$ fingers for $B, 3$ fingers for $C$ and 4 for $D$ ".

1. Notice how the learners respond. Ask a learner who gave answer A to explain why he or she gave that answer. DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
2. It is important for learners to explain the reasons for their answers. Putting thoughts into words may help them to gain better understanding and improve their communication skills.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. Ask the class to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.
The correct answer is: B for Quiz 1
and $\mathbf{D}$ for Quiz 2
https://diagnosticquestions.com


## Follow up

30-minute Fractals Lesson and follow-up
https://aiminghigh.aimssec.ac.za/years-2-to-10-30-minute-fractals-lesson-and-follow-up/
Squareflake Fractal
https://aiminghigh.aimssec.ac.za/years-7-to-12-squareflake-fractal/
Make a von Koch Poster
https://aiminghigh.aimssec.ac.za/years-4-10-make-a-von-koch-poster/
Pascal's triangle and Fractal Patterns
https://aiminghigh.aimssec.ac.za/years-9-to-12-pascals-triangle-and-fractal-patterns/

| MIATHS | Go to the AIMSSEC AIMING HIGH website for lesson ideas, solutions and <br> curriculum links: http://aiminghigh.aimssec.ac.za <br> Subscribe to the MATHS TOYS YouTube Channel |
| :--- | :--- |
| TOYS | https://www.youtube.com/c/mathstoys <br> Download the whole AIMSSEC collection of resources to use offline with <br> the AIMSSEC App see https://aimssec.app Find the App on Google Play. |


| Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and school years up to Secondary 5 in East Africa. <br> New material will be added for Secondary 6. <br> For resources for teaching A level mathematics (Years 12 and 13) see https://nrich.maths.org/12339 <br> Mathematics taught in Year 13 (UK) \& Secondary 6 (East Africa) is beyond the SA CAPS curriculum for Grade 12 |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | Lower Primary Approx. Age 5 to 8 | Upper Primary Age 8 to 11 | Lower Secondary Age 11 to 15 | Upper Secondary Age 15+ |
| South Africa | Grades R and 1 to 3 | Grades 4 to 6 | Grades 7 to 9 | Grades 10 to 12 |
| East Africa | Nursery and Primary 1 to 3 | Primary 4 to 6 | Secondary 1 to 3 | Secondary 4 to 6 |
| USA | Kindergarten and G1 to 3 | Grades 4 to 6 | Grades 7 to 9 | Grades 10 to 12 |
| UK | Reception and Years 1 to 3 | Years 4 to 6 | Years 7 to 9 | Years 10 to 13 |

