AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

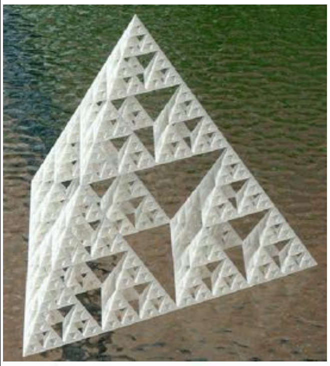
SIERPISKI NUMBER AND SHAPE PATTERNSN

The question is how many little tetrahedra, like the blue model shown, were used to make this 6.5 metre high red balloon model tetrahedron. It was built in a shopping mall in Cambridge, UK to set a Guinness World Record.

The smallest tetrahedron, the blue model (Stage 0) is made from 6 balloons, each 25 centimetres long.

In a perfect model the green (Stage 1 – 50cm) tetrahedron, made from four 25 cm pyramids, would have edges of length 50 centimetres.

Then 4 of the Stage 1 tetrahedra are used to make the Stage 2 – 1 metre model, and 4 of those to make the next one Stage 3 – 2 metres, and so on, and so on. At each stage of the construction 4 tetrahedra are used to make a bigger tetrahedron and the lengths of the edges are double the lengths of the edges at the previous stage.



This white 3D printed model shows the construction of the red balloon model.

The red balloon model was made from 1024 small tetrahedra with edges of length 25 centimetres. Clearly it was not a perfect shape like the white model, but if it had been then the edges would have measured 8 metres.

The rainbow tetrahedron is a Stage 4 – 4 metre construction made with 4 of the Stage 3 – 2 metre tetrahedra.

There are many other questions about number patterns and geometry that you can investigate based on this structure. For example: ‘How many triangular faces at each stage?’ ‘If the blue (Stage 0) tetrahedron were solid rather than skeletal, what fraction of the volume of the tetrahedron is filled at each stage and what fraction is empty space?’ ‘What shape is the hole in the middle?’ Let us know what you find out.

HELP SIERPINSKI TRIANGLE (The same construction in 2 dimensions)

Use triangular grid-paper if possible and make your fractal as big as possible.

Step One Draw an equilateral triangle with sides of 2 triangle lengths each. Connect the midpoints of each side. How many equilateral triangles do you now have?

Shade the three outer triangles. Alternatively, think of this as cutting a triangular hole in the triangle.

Step Two Put three of these patterned triangles together. Alternatively draw another equilateral triangle with sides of 4 triangle lengths each and mark the edges into 4 equal lengths. Shade the nine small triangles as shown. Note the larger 'hole' and the three smaller 'holes'. All the diagrams below are scaled down to reduce them in size.

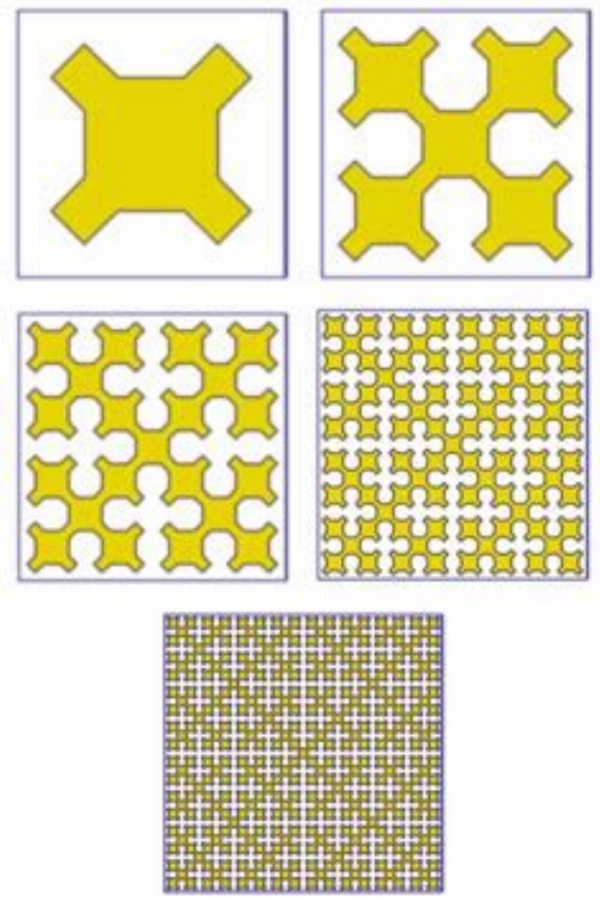
Step Three Put three of these patterned triangles together, that is nine of the patterned triangles from Step One. Alternatively draw an equilateral triangle with sides of 8 triangle lengths. Shade the 27 small triangles as shown. You will have 1 large, 3 medium, and 9 small 'holes'. Note the number pattern 1, 3, 9, 27,...

Step How about making a poster? If each member of your class makes one of the patterned triangles from Step Three, then 3 of them can be assembled to make a patterned triangle as in the diagram below, or 9 or 27 (or even 81...) can be assembled to make patterns getting closer to a Sierpinski Triangle Fractal. Use your artistic creativity to shade the triangles in interesting colour patterns.

NEXT

To develop your investigative and communication skills, choose and investigate one aspect of the construction of a giant Sierpinski Tetrahedron and write up your findings. Different learners could pursue different investigations to answer different questions (see the list in the 'Why do this activity' section). A project like this offers the opportunity for you to prepare a report, to make your own models and posters and give a talk to present your discoveries to the whole class.

One choice of project could be to find out more about Waclaw Sierpinski (1882 -1969) and his work. He was born in Warsaw, Poland when it was part of the Russian Empire. He was a mathematician, specializing in set theory, topology and number theory who experienced the hardships of the two World Wars. He wrote 724 research papers and 50 books. Warsaw University Library and Sierpinski's own house, including his library and personal papers, were burnt down by the Nazis and many of his mathematician colleagues and friends were killed. He retired in 1960 at the age of 78 but continued with his editorial work and giving lectures.

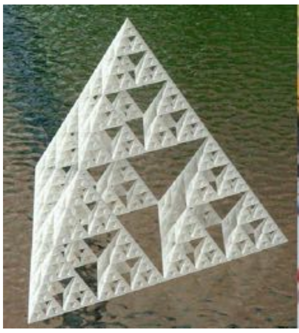


This shows 4 stages of Sierpinski's space filling curve. In the limit the fractal curve passes through every interior point of the square. However, the area enclosed by the curve is less than half the area of the square.

NOTES FOR TEACHERS

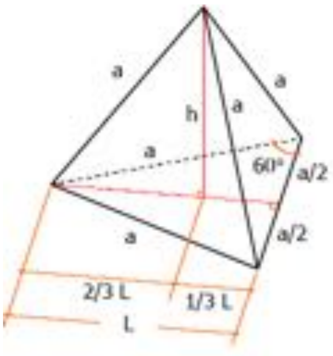
SOLUTIONS

Here are some answers to help teachers to use this activity as a starting point for mathematical many different activities and investigations. There is scope at all levels from developing language; scope for young learners just describing what they see; and for upper secondary level to work on calculations involving scaling, trigonometry, geometric series and ideas of limits and infinity.

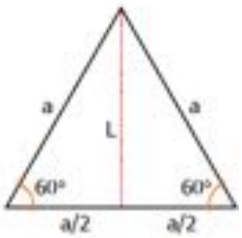


The photo shows the Guinness World Record Breaking Balloon Model constructed in a shopping mall in Cambridge UK in March 2014 as a fundraiser for AIMSSEC, by a team led by Bubblz the Mathematical Clown

Clearly it was not a perfect shape like the white one shown beside it.



Regular Tetrahedron



One Face of Regular Tetrahedron

These diagrams should help you to calculate the dimensions in the 'ideal' Sierpinski Tetrahedron at the various stages.

Note that a fractal is infinite so our models are just a few stages of a fractal.

STAGE	0	1	2	3	4	5	6
Number of tetrahedra	1	4	4^2=16	4^3=64	4^4=256	4^5=1024	4^6=4096
Number of 25 cm balloons	6	24	96	384	1536	6144	24576
							Target for new record
Edge length a metres	0.25	0.5	1	2	4	8	16
Altitude of triangular face L = a√3/2 metres	0.217	0.433	0.866	1.732	3.464	6.928	13.856
Vertical height H = a√(2/3) metres by Pythagoras Th.	0.204	0.408	0.816	1.633	3.266	6.532	13.064
Vertical height in feet	0.670	1.339	2.679	5.358	10.715	21.430	42.460

DIAGNOSTIC ASSESSMENT This should take about 5–10 minutes.

Choose the Quiz most suitable for your class, write the question on the board, say to the class:

"Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".

1. Notice how the learners respond. Ask a learner who gave answer A to explain why he or she gave that answer. DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
2. It is important for learners to explain the reasons for their answers. Putting thoughts into words may help them to gain better understanding and improve their communication skills.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. Ask the class to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.

The correct answer is: **B** for Quiz 1
and **D** for Quiz 2
https://diagnosticquestions.com

Why do this activity?

This activity offers motivation and scope for mathematical work at all levels:

1. **Language development – all ages**. Ask 'What do you see? Can you describe how the model is made?
2. **Number patterns – all ages**. 1, 2, 4, 8, 16, ... also 1, 4, 16, 64, ... 1, 3, 9, 27,... and 6, 24, 96, ...
3. **Ideas of infinite processes and limits**. Ask "What if this process continued on and on getting bigger and bigger?' "Can you imagine this process continued on and on getting smaller and smaller?'
4. For the older students: **summing geometric series**, both with terms getting larger and also with terms getting smaller.
5. This leads to **understanding fractals** and links with other fractal structures and fractals in nature.
6. Recognition and **properties of equilateral triangles and regular tetrahedra**.
7. **3D geometry**, **measures and calculation of lengths and areas**. Applications of Pythagoras Theorem.
8. **Similarity, enlargement and scale factors**. For example: If the small Stage 0 tetrahedra are solid (rather than skeletal) how many times greater would be the volume of the record breaking tetrahedron be compared to the volume of the original small tetrahedron? Answer: the volume scale factor is $(2^3)^5=262144$
9. There is a big hollow space inside each Sierpinski Tetrahedron. What is the shape of that space? It is certainly not another tetrahedron. Have a look and see for yourself.
10. Use of a spreadsheet for the calculations.

Learning objectives

In doing this activity students will have an opportunity to:

- investigate Number Patterns, for example: 1, 2, 4, 8, … … and 6, 24, 96, also 1, 4, 16,…. and 1, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, … and 1, 3, 9, 27,… and 1, $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}$,… (all ages);
- work on summing geometric series (for the older students);
- gain some understanding of infinite processes and limits;
- develop the visualization skills to recognize equilateral triangles and regular tetrahedra when viewed from different angles and when 3D structures are represented in 2D;
- develop understanding of properties of 2D and 3D shapes;
- work on 3D geometry, measures and calculation of lengths, areas and volumes including applications of Pythagoras Theorem;
- gain better understanding of similarity, enlargement and scale factors.

Generic competences

In doing this activity students will have an opportunity to:

- develop language and communication skills;
- develop geometric thinking and the ability to reason from pictures and diagrams;
- gain some understanding of fractals and the links with other fractal structures and fractals in nature;
- gain better understanding of similarity, enlargement and scale factors as used in map scales and design drawings;
- develop skills in using Excel spreadsheets for calculations.

Suggestions for teaching

Teachers must be selective because there is a wealth of mathematical content here for all ages. They must decide on the **learning objective** for their lesson and focus the lesson accordingly.

There is considerable advantage in coming back to this same Sierpinski Model on different occasions to pursue different learning objectives in different lessons.

Your learning objective might be the development of investigative and communication skills. Then this activity could be the basis of mathematical project work. Different learners could pursue different investigations to answer different questions (see the list above in the 'Why do…' section). A project like this offers the opportunity for learners to prepare reports, models, posters and talks to present to the whole class.

You might like to start with the Sierpinski Tetrahedron (below) or the Sierpinski Triangle activity (See HELP section).

Learners could make their own model Sierpinski tetrahedron in order to understand the repetitive process of construction and to help them to visualize the patterns. There are many materials you could use: toothpicks fixed together using plasticine, Prestik or Blu Tack, or rolled paper sticks tied together by string, or drinking straws and pipe cleaners, or wooden skewers and rubber bands…

Make four tetrahedra. Fix 3 tetrahedra together to make a larger triangular base and stick the fourth tetrahedron on top.

It is not necessary to go any further than this first stage of the process but the class could make a bigger model by putting together models made by different learners.

QUESTIONS FOR MATHS LESSONS

1. Look at the second triangle in Step 1 in the HELP section. What fraction of the triangle is shaded? What fraction is unshaded?
2. You can think of the diagrams of the Sierpinski Triangle on page 2 in two different ways. As drawn they show a process of making more triangular holes at smaller and smaller scales, in stages that, carried on forever would produce the Sierpinski fractal. Alternatively these diagrams of the Sierpinski Triangle could have been reduced in scale to fit more of models that were growing in size, into the same actual space. What are the linear scale factors in these reductions? What are the area scale factors?
3. What fractions of the triangle are shaded in Steps 2, 3 and 4? What fractions are unshaded?
4. Do you see a pattern here? Use the pattern to predict the fraction of the triangle you would shade in a poster made with 9 of the patterned triangles from Step 3?
5. Confirm your prediction and explain it.
6. What fraction is shaded in a poster made with 27 of the patterned triangles from Step 3?
7. Write the fractions of the triangles shaded in the above steps in order from least to greatest. Write a statement about how their order connects to the shading?
8. **CHALLENGE 2D:** Develop a rule or formula so that you could calculate the fraction of the area which is shaded for any step.
9. Find another interesting pattern in the Sierpinski Triangle. Write a paragraph describing your pattern.
10. **CHALLENGE 3D:** If the Scale 0 tetrahedra were solid (rather than skeletal) how many times greater would be the volume of the record breaking tetrahedron compared to the volume of the original small tetrahedron? Answer: the volume scale factor is $(2^5)^3 = 32768$

11. There is a big hollow space inside each Sierpinski Tetrahedron. What is the shape of that space? It is certainly not another tetrahedron. Have a look and see for yourself.

Key questions

- What do you see?
- Can you describe the shapes you see in the model?
- Can you describe how the model is made?
- Can you find similar shapes?
- How many … (edges, tetrahedral) at each stage?
- What are the edge lengths at each stage?

Follow up

30-minute Fractals Lesson and follow-up
https://aiminghigh.aimssec.ac.za/years-2-to-10-30-minute-fractals-lesson-and-follow-up/
Squareflake Fractal
https://aiminghigh.aimssec.ac.za/years-7-to-12-squareflake-fractal/
Make a von Koch Poster
https://aiminghigh.aimssec.ac.za/years-4-10-make-a-von-koch-poster/
Pascal's triangle and Fractal Patterns
https://aiminghigh.aimssec.ac.za/years-9-to-12-pascals-triangle-and-fractal-patterns/

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and school years up to Secondary 5 in East Africa.
New material will be added for Secondary 6.
For resources for teaching A level mathematics (Years 12 and 13) see https://nrich.maths.org/12339
Mathematics taught in Year 13 (UK) & Secondary 6 (East Africa) is beyond the SA CAPS curriculum for Grade 12

	Lower Primary Approx. Age 5 to 8	Upper Primary Age 8 to 11	Lower Secondary Age 11 to 15	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13