Why do some fractions have recurring decimal expansions with patterns of nines?

A fraction n with decimal expansion of the special periodic form

$n = 0.a_1a_2a_3...a_kb_1b_2b_3...b_k$ *recurring*

where $(a_1a_2a_3...a_k) + (b_1b_2b_3...b_k) = 9 ... 9 = 10^k - 1$

is given by the formula $n = \frac{a_1a_2a_3...ak + 1}{10^k + 1}$.

For example, for $k = 4$,

suppose $a_1a_2a_3a_4 = 3251$ then $\frac{3252}{10^4+1} = \frac{3252}{10001}$

$= 0.32516748...$

and $3251 + 6748 = 9999$

This can be proved using the sum of geometric series.

Proof Sorter activity 'Recurring decimals with patterns of nines'.

Cut out the strips and put them in the correct order to give a proof of the theorem.

Theorem 1. *Suppose n has a decimal expansion of the special periodic form*

$$n = 0{\cdot}\overline{a_1 \cdots a_k b_1 \cdots b_k}$$

where $a_1 \cdots a_k + b_1 \cdots b_k = 9 \cdots 9 = 10^k - 1$. *Then*

$$n = \frac{a_1 \cdots a_k + 1}{10^k + 1}$$

$$= \frac{10^k a_1 \cdots a_k + [(10^k - 1) - a_1 \cdots a_k]}{10^{2k} - 1}$$

Proof. We are given that $n = 0{\cdot}\overline{a_1 \cdots a_k b_1 \cdots b_k}$ so, writing this recurring decimal as an infinite series, we have

$$n = \frac{a_1 \cdots a_k}{10^k}\left(1 + \frac{1}{10^{2k}} + \frac{1}{10^{4k}} + \cdots\right) + \frac{b_1 \cdots b_k}{10^k}\left(1 + \frac{1}{10^{2k}} + \frac{1}{10^{4k}} + \cdots\right)$$

$$= \frac{a_1 \cdots a_k + 1}{10^k + 1}.$$

$$= \frac{(10^k - 1)(1 + a_1 \cdots a_k)}{(10^k - 1)(10^k + 1)}$$

$$= \left(\frac{10^k a_1 \cdots a_k + b_1 \cdots b_k}{10^{2k}}\right)\left(\frac{1}{1 - 1/10^{2k}}\right)$$

For the solution see the Magic Numbers Inclusion and Home Learning Guide.
https://aiminghigh.aimssec.ac.za/magic-numbers/