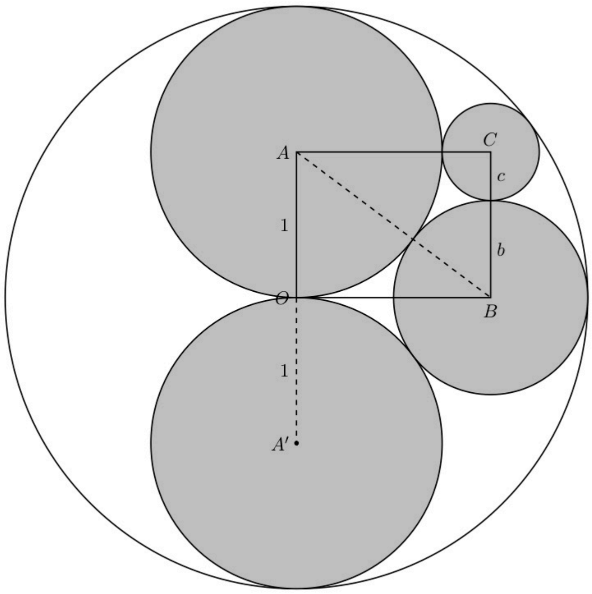
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

KISSING CIRCLES



The outer circle, centre O, has radius 2 units.
The points O, A, A′, B and C are centres of circles that are tangent to each other where they touch.

It can be shown that OACB is a rectangle.

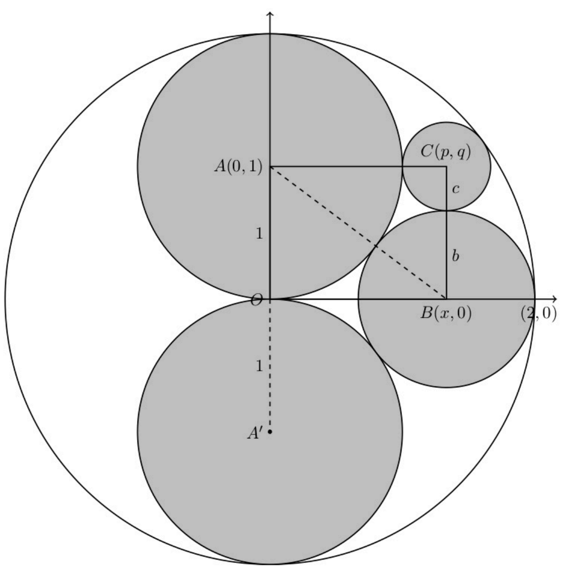
1. Prove that OA⊥OB
2. Find the radii b and c and the ratio of the radii of the circles with centres A, B and C.
3. What do you notice about triangle OAB?

HELP

Where the circles touch they have a common tangent and each radius is perpendicular to the tangent. What does this tell you about the lines joining the centres of the circles?

NEXT – MORE CHALLENGING

The proof that OACB is a rectangle can be done using coordinate geometry, the formula for the distance between two points and the fact that the circles all touch each other.



1. Prove that OA⊥O. Then use Pythagoras Theorem for Δ AOB and the fact that the circle centre B touches the outer circle to show that $b = \frac{2}{3}$.

2. The circle centre C touches 3 other circles, use this:

a) Write down 3 equations involving p, q and c and, from them, 3 expressions equal to $p^2 + q^2 - c^2$.

b) Find 3 linear expressions involving p, q and c.

c) Find an expression giving p in terms of c.

d) Find an expression giving q in terms of c.

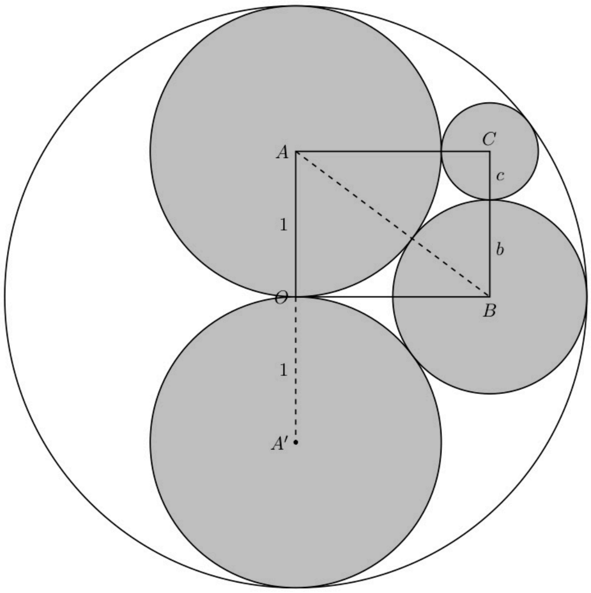
e) Eliminate p and q to produce a quadratic equation in c and solve this equation to find c.

3. For each value of c, find p and q and identify the two circles to which these values correspond.

4. Explain why the values found show that OACB is a rectangle.

NOTES FOR TEACHERS

SOLUTION

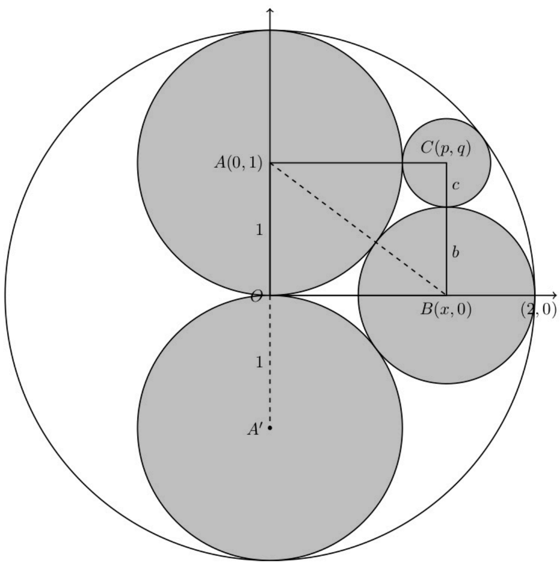


1. OB is tangent to the circle centre A by symmetry and the tangent is perpendicular to the radius so OA⊥OB.
2. Where the circles touch they have a common tangent, and each radius is perpendicular to the tangent. This means that the lines joining the centres of the circles go through the point of tangency.

Assuming OACB is a rectangle this gives:
OA = BC = b + c = 1
OB = AC = 1 + c
AB = 1 + b
By Pythagoras Theorem $AB^2 = OA^2 + OB^2$ so
$(1 + b)^2 = 1^2 + (1 + c)^2$ but $b + c = 1$ and this gives
$(1 + b)^2 = 1 + (2 - b)^2$ which has the solution $b = \frac{2}{3}$.
So $a = 1$, $b = \frac{2}{3}$ and c= $\frac{1}{3}$.
The ratio $a : b : c = 3 : 2 : 1$

3. Triangle OAB has edges 1, $\frac{4}{3}$, $\frac{5}{3}$.
It is a 3 – 4 – 5 triangle.

NEXT EXTENSION



1.The proof that that OA⊥OB.is as given above.
Pythagoras Theorem for Δ AOB gives $1 + x^2 = (1 + b)^2$ and the fact that the circle centre B touches the outer circle gives $x + b = 2$
So $1 + (2 - b)^2 = (1 + b)^2$ gives $b = \frac{2}{3}$ and $x = \frac{4}{3}$.

3.a) From OP: $\sqrt{(p^2 + q^2)} + c = 2$ [1]

From BC: $(p - \frac{4}{3})^2 + q^2 = (\frac{2}{3} + c)^2$ [2]

From AC: $(p^2 + (q-1)^2) = (1 + c)^2$ [3]

From [1] $(p^2 + q^2 - c^2) = 4 - 4c$

From [2] $(p^2 + q^2 - c^2) = \frac{8}{3}p - \frac{16}{9} + \frac{4}{9} + \frac{4c}{3}$

$= \frac{8}{3}p - \frac{4}{3} + \frac{4c}{3}$

From [3] $(p^2 + q^2 - c^2) = 2q - 1 + 1 + 2c = 2q + 2c$

From [1], [2] and [3]: $p = 2 - 2c$ and $q = 2 - 3c$
so $p^2 + q^2 = (2 - 2c)^2 + (2 - 3c)^2 = (2 - c)^2$.
This gives the quadratic equation for c: $3c^2 - 4c + 1 = 0$

$(c - 1)(3c - 1) = 0$

So $c = \frac{1}{3}$, $p = \frac{4}{3}$, $q = 1$ corresponding to the circle centre C radius $c = \frac{1}{3}$ unit.
or $c = 1$, $p = 0$, $q = -1$ corresponding to the circle centre A′ radius 1 unit.

4.The coordinates found for centre C $(p, q) = (\frac{4}{3}, 1)$ show that CA⊥ CB & OACB is a rectangle.

DIAGNOSTIC ASSESSMENT

This should take about 5–10 minutes. It can be used before or after the lesson.

Write the question on the board, say to the class:

"Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".

Which of the following statements is false?

A. A tangent to a circle is perpendicular to the radius at the point of contact.
B. A tangent to a circle cuts the circle in two coincident points.
C. Two tangents to a circle from a point outside the circle can be different lengths
D. The angle between a tangent to a circle and a cord from the point of contact is equal to the angle subtended by that chord at the circumference of the circle.

1.Notice how the learners respond. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer. It is important for learners to explain the reason for their answer.

2.Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.

3.Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.

The correct answer is: C. This statement is false. https://diagnosticquestions.com

Why do this activity?

The activity gives students experience of applying what they know about circle geometry to solve a problem. It involves application of the fact that a tangent to a circle is perpendicular to the radius at the point of contact. The NEXT extension problem involves Analytic Geometry and provides an extra challenge for older students who are enthusiastic about tackling challenges in mathematics.

Learning objectives

In doing this activity students will have an opportunity to review circle geometry.

Generic competences

In doing this activity students will have an opportunity to **r**eason logically and construct an argument giving evidence for all the statements that they make.

Suggestions for teaching

Start with the Diagnostic Quiz and a review of what the students know about Circle Geometry.

If possible give out copies of the diagram on page 1, or copy the question on the chalkboard.

The students should work individually and then work in pairs and compare and check their answers and arguments. Later in the lesson, pairs should work with a second pair to compare and check their answers.

The first question, assuming that OACB is a rectangle, is straightforward and any Year 12 or 13 student should be able to do this. The second part of the question (the NEXT challenge) does **not** require advanced mathematics (beyond the level taught in school) but it does involve a lot of algebra, in particular 3 simultaneous equations involving quadratic expressions. Solving 3 simultaneous equations requires care and perseverance, but students who relish a challenge can get a lot of satisfaction from working through the algebra and getting the final result.

Then ask representatives to present their work to the class. Finally summarize what has been learned.

Key questions

- Those two circles touch. What do you know about the line joining the centres of the circles?
- Can you find the lengths of the edges of that triangle?
- You know that is a rectangle. What can you say about the lengths of the edges?
- Your ratio involves fractions. How would you express the same ratio in whole numbers?
- Imagine extending the line OB to the edge of the outer circle. What could you deduce from that?

Follow up

Investigating Circle Theorems
https://aiminghigh.aimssec.ac.za/years-10-11-investigating-circle-theorems/

Circle inscribed in a Quadrilateral
https://aiminghigh.aimssec.ac.za/years-10-12-circle-inscribed-in-quadrilateral/

Cyclic https://aiminghigh.aimssec.ac.za/years-11-12-cyclic/

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. New material will be added for Secondary 6.
For resources for teaching A level mathematics see https://nrich.maths.org/12339
Note: The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is beyond the school curriculum for Grade 12 SA.

	Lower Primary Age 5 to 9	Upper Primary Age 9 to 11	Lower Secondary Age 11 to 15	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6