TRI-FOLD Inclusion and Home Learning Guide is part of a Learning Pack downloadable from the AIMING HIGH website
https://aiminghigh.aimssec.ac.za/tri-fold/
It provides related activities for home learning for all ages and learning stages from pre-school to school-leaving, together with guidance for home-learning and also for inclusion in school lessons, all on the Common Theme **PAPER FOLDING**
    Guidance for school lessons is given in the Notes for Teachers documents.
Choose what seems suitable for the age or attainment level of your learners.

TRI-FOLD
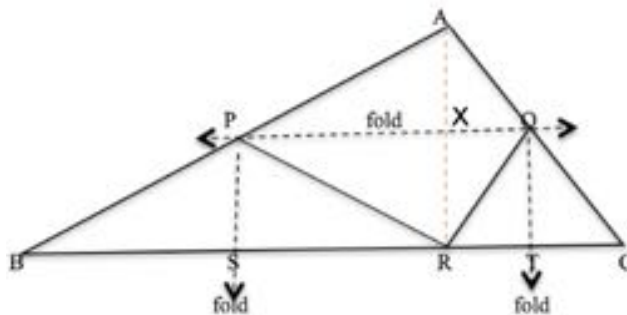
Cut out two identical copies of a triangle that has no equal angles and no right angle. Label the triangle ABC with A the largest angle.

N.B. When you mark points, it will make the folding easier if you mark points on both sides of the paper.

Method 1

a. To find the midpoint P of edge AB, fold vertex A to meet vertex B, gently make a small crease (not a fold along a line) and label the point on the crease P. Do the same to find the midpoint Q of edge AC.

b. Fold along PQ. You should find that vertex A just touches the baseline BC. Mark this point and label it R on both sides of the paper. Open the triangle out. Fold along AR. What do you notice about ∠ARB and ∠ARC?

c. Open the triangle out. Fold B to coincide with R and label point S. What do you notice?

d. Fold C to coincide with R and label point T. What do you notice?

e. Fold along PQ again and you will find that points A, B and C all coincide with R. What do you notice?

f. Does the folding experiment tell you anything else about triangles? Why does all this happen? Important geometric facts are demonstrated by this paper folding activity. Will the same happen for all triangles? If so, can you prove it?

Method 2

a. Take the second copy of triangle ABC. To find the point R, the foot of the perpendicular from A to BC, firmly fold through point A such that the edge BC is folded back on itself with point C lying in a new position on the edge BC and mark the point R. Flatten the triangle and mark the line AR. What do you notice about the angles ARB and ARC? Will this happen for all triangles? If so, can you prove it?

b. To find the point S, the midpoint of BR, fold the edge BC back on itself again but this time make B coincide with R. Mark the point S where this fold cuts edge BC

and the point P_1 where the fold cuts edge AB. What do you notice about the angles P_1SB and P_1SR? Check that P_1 is actually the midpoint P of AB by folding vertex A to meet vertex B as in Method 1 so that AP = PB, and gently making a small crease at P (not a complete fold).

c. Similarly, to find the point T, the midpoint of RC, fold the edge BC back on itself again but this time make C coincide with R. Mark the point T where this fold cuts edge BC and the point Q_1 where the fold cuts edge AC. What do you notice about the angles Q_1TR and Q_1TC? Check that Q_1 is actually the midpoint Q of AC by folding vertex A to meet vertex C as in Method 1 so that AQ = QC, and gently making a small crease at Q (not a complete fold).

d. Fold along PQ and you will find that points A, B and C all coincide with R. What do you notice?

e. Compare the final result you obtained by the two methods.

f. Talk about what you have discovered with other members of your group. If they worked with triangles of different shapes did everyone find the same geometric properties for their triangle?

HELP

It will help you to see which angles are equal, and other geometric properties, if you use colours to mark each set of equal angles in a different colour, and each set of equal lengths in a different colour.

NEXT

This activity, because it is so open ended, comes with its own extensions. It will take a long time to explore all the possibilities. Can you answer all the Key Questions fully?

Resources: paper, scissors, rulers, pencils and glue.

LESSON STARTER FOR ALL AGES

The Tri-Fold activity Method 1 as described above.

Let each participant make their own triangle.

Very young learners can work with an older partner.

After folding everyone should get a result like this illustration that can be labelled with the points and angles as shown.

The participants should talk about what they notice.

Further work for different age groups is described below.
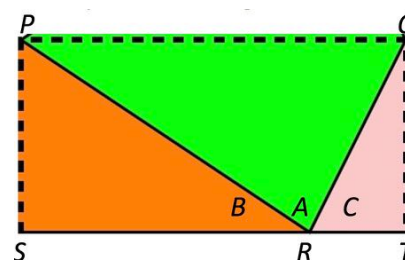
HOME LEARNING AND INCLUSION GUIDE

THEME: PAPER FOLDING

Suggestions for Home Learning

FOR ALL AGE-GROUPS:- the suggested approach is to ask very open questions. Such questions enable learners to think independently, to notice things for themselves and to suggest properties and ideas that they could follow up. If your learners are accustomed to this style of teaching they'll respond well. For this activity you could ask the following questions:

- What do you notice?
- Why does this happen?
- Will it happen for all triangles? Why or why not?
- Does the folding experiment tell you anything else about triangles?

The activity on page 1 demonstrates at least 4 geometrical topics.

1. The angles of a triangle add up to 180°.
2. The area of a triangle is half the base times the height.
3. Properties of reflections and enlargements.
4. The line joining the midpoints of the edges of a triangle is parallel to the third edge and half the length. This is called the **triangle midpoint theorem**, it can be proved formally and statements 1, 2 and 3 can be derived from it.

Why do this activity?

This **activity** is an ideal **inclusion resource for schools and also an ideal** activity for a multi-age group of young people learning together at home**.** Mature adults, 18-year olds and young learners will be equally impressed by what is revealed by this simple paper-folding activity. The activity uses tactile senses and visualisation to suit different individual learning styles. It also provides links between different geometrical properties in a way that is likely to be remembered.

In a home learning group, younger children may notice aspects that older ones have not spotted. Hearing what an older learner says may encourage younger learners, even if they don't understand everything at the time. If there is the opportunity to submit written work to the child's school-teacher, then the teacher can make sure errors are noticed and corrected.

This activity, because it is so open ended, comes with its own extensions. Very few learners will have explored all the possibilities. They should be congratulated on the good work they have already done, and then asked more questions to help them to find more possibilities for themselves. For example, it will take some time, and some thought, to answer all the key questions (see page 11) fully.

This activity is a simple practical experiment with a surprising and pleasing result. Why do the 3 vertices of the triangle meet exactly on the baseline? Was this just lucky or will it always happen. The resources cost nothing and many more geometrical properties come out of this activity.

It brings together concepts of transformation geometry (which is very important in higher mathematics) with properties that can be stated in terms of similar triangles, isosceles triangles, congruent triangles, parallel lines etc.

Four topic areas in geometry are demonstrated. The activity can be used as an introduction to any one of the topics by asking questions to draw the learners' attention to particular features and geometrical properties so that the learners notice properties relating to the chosen focus.

For older learners the activity is also good as a review of geometry that the learners should already know. Because it is very open ended it is ideal for including all abilities. Lower attaining learners may not notice all the properties but they should be able to succeed with the folding, and (with suitable help) be able to understand the proof of at least one of the results. There is plenty of challenge for higher attaining learners and scope to think mathematically and prove more results.

It is helpful or everyone to finish by making their own summary of what has been learned.

Early Years
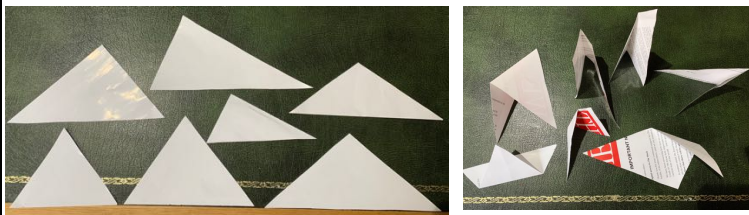
Cut out some triangles.

Make **one** fold anywhere in the triangle. What happens?

What shapes can you make?

One photo shows some triangles on my desk that I cut from the page of a magazine, and the other photo shows the same triangles each with one fold.

Can you make a fold that produces 2 triangles?

Can you make a fold that produces 1 triangle and 1 quad (short for quadrilateral)?

What do you notice about your shapes?

Play with your folded paper triangles. Talk about what you notice about them. Perhaps colour them and stick them on a poster to make a picture.

Grown-ups: this should be free play with some talk about the shapes and certainly no formal teaching. The name quad (short for quadrilateral) can be used for 4-sided shapes and some children will want to learn the full name.

Lower Primary

Cut out some triangles.

Make **one** fold anywhere in the triangle. What happens? What shapes can you make?

Make **two** fold anywhere in the triangle. What happens? What shapes can you make?

One photo shows some triangles on my desk that I cut from the page of a magazine. The other two photos show the same triangles with one fold and with two folds.

Can you make fold that produces 2 triangles? What about 3 triangles?

What other shapes can you make? What do you notice about your shapes?

Play with your folded paper triangles. Talk about what you notice about them.

Perhaps colour them and stick them on a poster to make a picture.

Grown-ups: this should be free play with some talk about the shapes and very little formal teaching. The name quad (short for quadrilateral) can be used for 4-sided shapes and some children will want to learn the full name.

You might talk about some of the special sorts of quadrilateral like squares and rectangles and kites and rhombuses and parallelograms and trapezia (trapezoids if you are American), but only talk about any of them if the children have actually made one of the special quadrilaterals by folding, and have noticed something special about a particular quadrilateral. Learning about the different sorts of triangle and different sorts of quadrilateral comes much later on when the children are older.
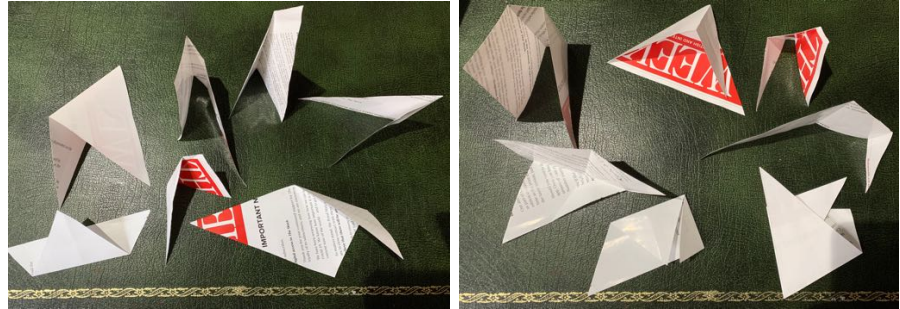
Upper Primary

If you have a group of children at home with a range of ages, and some 9 or 10 year olds or older, do the Tri-Fold paper-folding activity together in one group using Method 1. After that, depending on how much time you want to spend on this activity, you can ask different learners to work on different topics, perhaps on the property they spotted. Tell them to be prepared to explain what they have done so that all the learners will see the many possible ideas that can be developed.

Do the Tri-Fold activity and focus on
Topic 1: The angles of a triangle add up to 180°
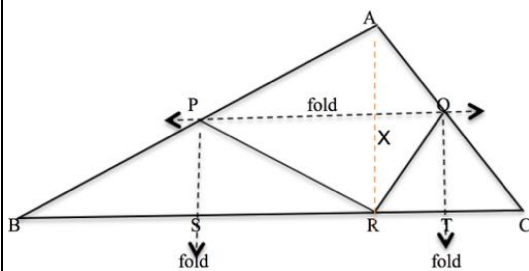
Draw the diagram and ask the learners to draw and cut out their own triangles, and to try to make them different from other people's triangles. Ask them to place the cut-out triangle so that the longest edge is the base and the biggest angle is at the top and to label their triangle ABC as in the diagram.

(*Note: labelling will have to be inside the cut-out triangle*)

Fold vertex A to meet vertex B to find the midpoint P of edge AB, gently make a small crease (not a fold along a line). Label the point P.

Fold vertex A to meet vertex C to find the midpoint Q of edge AC, gently make a small crease (not a fold along a line) and label the point Q.

Fold along PQ. You should find that vertex A just touches the baseline BC at point R.

Open the triangle out. Fold along AR. What do you notice about ∠ARB and ∠ARC?

Open the triangle out. Fold B to coincide with R and label point S. What do you notice?

Fold C to coincide with R and label point T. What do you notice?

Then ask the learners to fold A to exactly coincide with R. If they do this carefully they will see that this fold is along PQ; that **all three vertices meet exactly at the single point** R; and that the three angles lie together along the baseline of the triangle.

Ask: What do you notice? Does the folding experiment tell you anything else about triangles?

Why does all this happen?

Important geometric facts are demonstrated by this paper folding activity.

Will the same happen for all triangles? If so, can you prove it?

With a younger class this is sufficient as a demonstration that the angles of a triangle add up to two right angles or 180°. You may choose to give the 12 year olds further work on calculating angles using this fact. Textbooks have many such examples.

You can ask the learners what else they notice. They might notice:

- the triangle can be folded into a rectangle;
- the folding shows that the area of the triangle is double the area of the rectangle;
- the line PQ looks parallel to the base of the triangle;
- P and Q are the midpoints of AB and AC.

Your response can be to engage the learners in checking these observations and discussing them. All this is demonstration and not proof that the properties hold in general.

Key questions - Always ask learners to give reasons for their answers.
- Which angles are 90°?
- Where is the foot of the perpendicular from A to BC?
- Which mirror images do you see?
- Which lengths are equal to each other?
- Which angles add up to 180 degrees?

For secondary learners it is helpful to do the Diagnostic Quiz.

DIAGNOSTIC ASSESSMENT This should take about 5–10 minutes.
Write the question on the board, say to the class:
"Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".

Niamh wants to build a path that is equidistant from the two streetlights shown on the right.

Which of the following constructions will she need?

A — A perpendicular bisector
B — An angle bisector
C — A circle
D — An equilateral triangle

1. Notice how the learners respond. Ask them to think of the reason they gave their answer and to listen to other learners and try to decide if their own answer was right or wrong.
2. Ask a learner who gave answer A to explain why he or she gave that answer. DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. It is important for learners to explain the reasons for their answers. Putting thoughts into words may help them to gain better understanding and improve their communication skills.

4. Then do the same for answers B, C and D.
5. Ask the class to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.
6. The concept is needed for the lesson to follow so explain the right answer or give a remedial task.
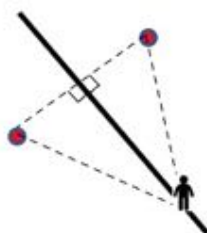
Correct answer A The path is the perpendicular bisector of the line between the two streetlights. The right-angled triangles in the diagram are congruent (RHS) so the hypotenuses of the triangles are equal and man on the path in the diagram is equidistant from the two streetlights.
**B. C. and D.** Learners who gave these answers may be guessing. They may vaguely remember doing constructions using circles to produce angle bisectors and equilateral triangles.   **https://diagnosticquestions.com**

Lower Secondary

Do the activity and focus on Topic 2: The area of a triangle is half the base times the height

You can use this activity as an introduction for a lesson on areas focussing on the formula for the area of a triangle. Then the class can do further work on areas.

Folding produces congruent triangles that are mirror images of each other. Asking learners to colour the mirror images draws attention to pairs of congruent triangles.
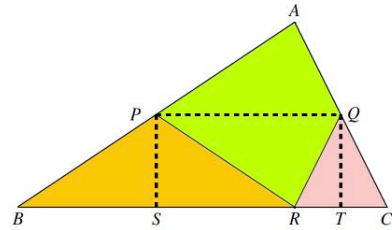
Notice that the height PS of the rectangle is half the height AR of the triangle and the base ST of the rectangle is half the base BC of the triangle.

Folding shows us that the area △ABC is twice the area of the rectangle.

It follows that area △ABC = 2 × ST × PS = 2 × $\frac{1}{2}$ base of triangle × $\frac{1}{2}$ height of triangle

This proves that the area of a triangle is half the base times the height of the triangle.

Noticing these geometric properties is a rich experience for young learners. Formal proofs will come later.

Key questions - *Always ask learners to give reasons for their answers.*
Which angles are 90°?
Tell me what you notice about the small triangles inside the big one.
Which pairs of triangles have the same area?
What shape did you make when you folded A, B and C to meet at R?
Can you see a rectangle in the picture?
Which triangles are in the rectangle?
Which triangles are not in the rectangle?
What does this show about the area of a triangle.

Years 9 and 10

Do the activity and focus on Topic 3: Properties of reflections and enlargements
Older learners should do the activities and notice for themselves the geometric facts that are demonstrated.

The fold lines are lines of symmetry (mirror lines) and △ABC is an enlargement of △APQ. In our lives everyone experience**s** reflections in mirrors and the enlargement of shapes on screens or as shadows on a wall. Talking about what they notice from this paper-folding activity lays a strong foundation for later work on geometry.

Key questions - Always ask learners to give reasons for their answers.
- Which angles are equal?
- Which lengths are equal?
- Which triangles are congruent?
- Which triangles are similar?
- Are there any enlargements are shown in the diagram? Tell me about them.
- What symmetries do you see?
- Are there any lines of symmetry? Tell me about them.
- Are there any reflections shown in the picture?

Years 11, 12 and 13

Do the activity and focus on Topic 4: The line joining the midpoints of the edges of a triangle is parallel to the third edge and half the length.

Older students must learn that demonstrations are not proofs. It is necessary to prove formally that the results are exactly what they seem, that their eyes are not deceiving them, and that these results always happen for all triangles. This activity can be used to give older students the experience of noticing geometrical properties for themselves, and of writing down their reasoning formally.

SOLUTION

1. Folding shows that the join of the midpoints of the edges of a triangle is parallel to the third edge and half the length of the third edge. This is proved below as a theorem.

2. Any triangle can be folded along 3 fold-lines as shown in the diagram, so that the vertices A, B and C meet at the point R on the edge BC. This brings the angles of the triangle together on the line BC showing that the angles of the triangle add up to 180°. The proof that the three vertices must come together at R follows from the theorem.

3. The folding activity shows that the area of the triangle is twice the area of the rectangle PQTS, that is half the base times the height. The proof also follows from the theorem.

**Theorem.** In the triangle $\triangle ABC$ below P is the midpoint of AB, and Q is the midpoint of AC

(thus AP = PB and AQ = QC). Then PQ = $\frac{1}{2}$ BC and PQ is parallel to BC.
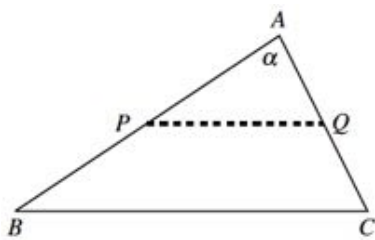
Proof. As AP/AB = AQ/AC = 1/2, and both $\triangle APQ$ and $\triangle ABC$ have the same angle α at A, we see that $\triangle APQ$ is similar to $\triangle ABC$. Therefore PQ/BC = 1/2; that is PQ = $\frac{1}{2}$ BC.

From similar triangles, we see that $\angle APQ = \angle ABC$; thus PQ is parallel to BC.

Corollary 1. The angles of a triangle add up to 180°.
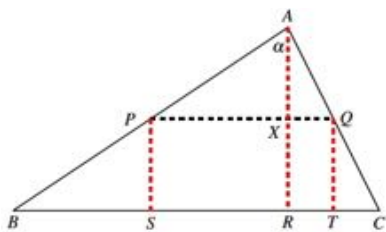
Proof.

Now consider the diagram where AP = PB and AQ = QC. We construct R, S and T so that the lines PS, AR and QT are perpendicular to BC (this can be done by folding the paper triangle).

As PQ is parallel to BC we see that $\triangle AXQ$ is similar to $\triangle ARC$, so that AX = $\frac{1}{2}$ AR, and XQ = $\frac{1}{2}$ RC.

As PQ is perpendicular to AR, and X is the midpoint of AR, it follows that A folds across PQ to R.

Also XQ = RT = $\frac{1}{2}$ RC and XR = QT (opposite sides of a rectangle) so C folds across QT to R.

The same is true for the other side of the triangle $\triangle ABC$, that is B folds across PS to point R.

So the 3 angles of the triangle are adjacent angles on a straight line and add up to 180°.

Corollary 2. The area of a triangle is half the base times the height.
Proof.

Triangles △PBS and △PRS are images by reflection in the fold line PQ, so these triangles are congruent and equal in area.

Triangles △QCT and △QRT are images by reflection in the fold line PQ, so these triangles are congruent and equal in area.

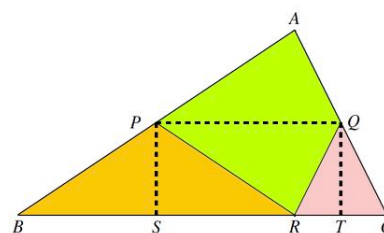Triangles △APQ and △RPQ are images by reflection in the fold line PQ, so these triangles are congruent and equal in area.

It follows that: area of triangle △ABC = 2 × area of rectangle PQTS.

BS = SR and RT = TC so ST=$\frac{1}{2}$BC

Also PS=$\frac{1}{2}$AR.

Area of the triangle △ABC = 2 × ST × PS = 2 × $\frac{1}{2}$BC × $\frac{1}{2}$AR = $\frac{1}{2}$ BC × AR, that is half the base times the height of the triangle.

Key questions - Always ask learners to give reasons for their answers.
- Which angles are 90°?
- Which angles are equal?
- What lengths are equal?
- Where is the foot of the perpendicular from A to BC?
- Which triangles are similar?
- Which triangles are isosceles?
- Which triangles are congruent?
- Can you see any length that is double another length?

Learning objectives

In doing this activity students will have an opportunity to explore and deepen their understanding of:
- the angles of a triangle add up to 2 right angles;
- the formula for the area of a triangle;
- reflection and enlargement;
- similar triangles and congruent triangles;
- the triangle mid-point theorem;
- how to recognize what is already known and construct a proof.

Generic competences

In doing this activity students will have an opportunity to:
- **think mathematically,** reason logically, give explanations and apply knowledge and skills;

- **develop visualization** skills to interpret or create images;
- **work in a team, share ideas and learn together**;
- **communicate** ideas and present information and ideas to others according the audience;
- to analyze, reason and record ideas effectively.

Follow up

Two by Two Puzzle https://aiminghigh.aimssec.ac.za/years-4-7-two-by-two-puzzle/
Wholesome Rectangles https://aiminghigh.aimssec.ac.za/years-5-8-wholesome-rectangles/
Is a Square a Rectangle https://aiminghigh.aimssec.ac.za/years-5-10-is-a-square-a-rectangle/
Angle sum: https://aiminghigh.aimssec.ac.za/years-8-10-angle-sum/
Sim Sets https://aiminghigh.aimssec.ac.za/grades-8-to-10-simsets/

<table>
<tr><td>**MATHS TOYS**</td><td>Go to the **AIMSSEC AIMING HIGH** website for lesson ideas, solutions and curriculum links: http://aiminghigh.aimssec.ac.za
Subscribe to the **MATHS TOYS YouTube Channel**
https://www.youtube.com/c/mathstoys
Download the whole AIMSSEC collection of resources to use offline with the **AIMSSEC App** see https://aimssec.app or find it on Google Play.</td></tr>
</table>

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and school years up to Secondary 5 in East Africa.
New material will be added for Secondary 6.
For resources for teaching A level mathematics (Years 12 and 13) see https://nrich.maths.org/12339
Mathematics taught in Year 13 (UK) & Secondary 6 (East Africa) is beyond the SA CAPS curriculum for Grade 12

	Lower Primary Approx. Age 5 to 8	Upper Primary Age 8 to 11	Lower Secondary Age 11 to 15	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13