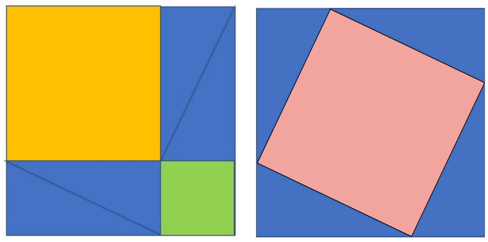
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

PYTHAGORAS JIGSAW Inclusion and Home Learning Guide is part of a double Learning Pack downloadable from the AIMING HIGH WEBSITE
https://aiminghigh.aimssec.ac.za/make-squares-jigsaw/ and
https://aiminghigh.aimssec.ac.za/pythagoras-jigsaw/
Together this pair of activities provides related activities on the Common Theme **SQUARES,** with guidance for home learning for all ages and learning stages from pre-school to school-leaving, together with resources for differentiation and inclusion in school lessons.
Choose what seems suitable for the age or attainment level of your learners.

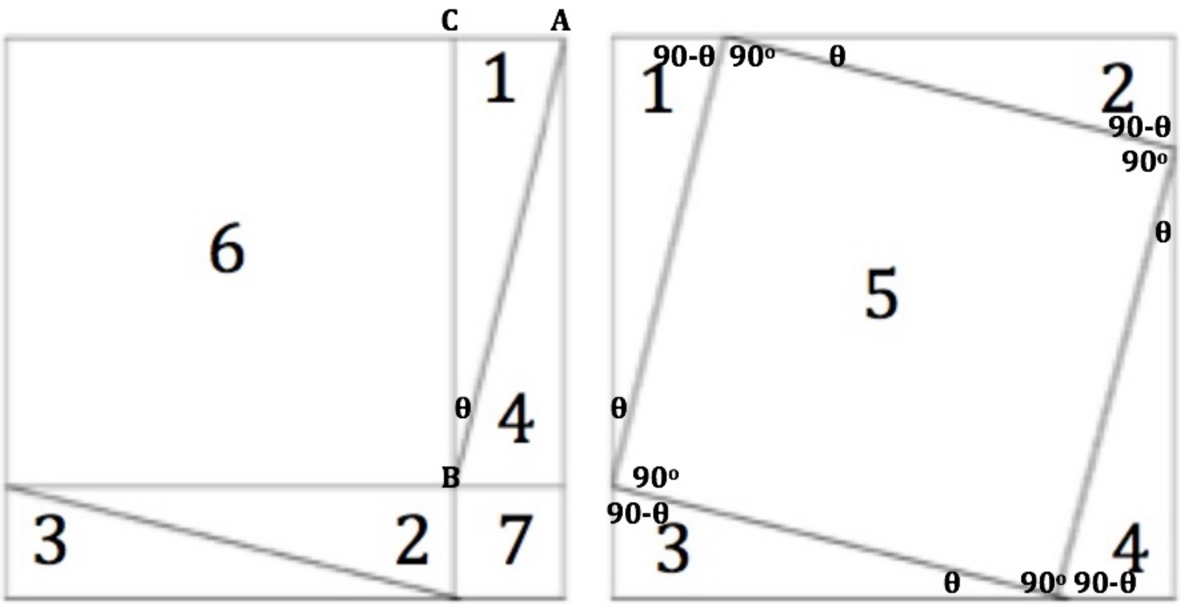
PYTHAGORAS JIGSAW



The diagrams show two different arrangements of pieces of the jigsaw inside a square frame. The right angled triangles numbered 1, 2, 3 and 4 are identical (congruent) copies of each other.

The Make Squares Jigsaw https://aiminghigh.aimssec.ac.za/make-squares-jigsaw/ has 7 pieces and these two solutions.

Here we refer to the edge lengths of the right angled triangle as a and b for the shorter edges and c for the hypotenuse. eople can make their own jigsaw pieces using any right angled triangle choosing their own values of a, b and c. The outer frame, into which both solutions fit, is a square of edge length $a + b$.



Think about the area inside the outer square frame. How is that area made up in the two different arrangements? How can you be sure that the shapes labelled 5, 6 and 7 are squares?

Just take away the four identical right angled triangles from each of the two arrangements. What is left? What do you notice? What is the same? What is different? What can you deduce from the areas?

What does this tell you about the area of square 5 compared with the areas of squares 6 and 7?

HELP

Cut out 4 congruent right angled triangles for yourself, choosing your own edge lengths, and arrange them in the first arrangement. Carefully draw the outline frame. Then arrange your 4 triangles in the frame in the second arrangement. Think about areas and answer the questions.

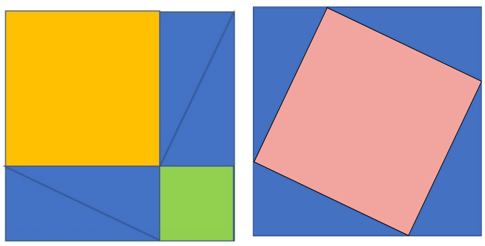
NEXT

1. Prove the algebraic identity $(a + b)^2 = a^2 + b^2 + 2ab$ using the areas in the diagram showing pieces 1, 2, 3, 4, 6 and 7.

2. (a) Choose your own values of lengths *a* and *b*. Carefully and accurately draw a square of edge length $a + b$ measuring the lengths you have chosen and the angles of 90^o. Make 4 identical copies of your chosen right angled triangle. Cut out the 4 triangles and fit them in the square frame in the two different arrangements as shown in the diagrams. Measure length c and, using *a* calculator, check that $a^2 + b^2 = c^2$.

   (b) Why could it be that the calculation may not show this result exactly?

   (c) If you are working with a group or a whole class, make a table on the board with 5 columns headed a, b, c, $a^2 + b^2$ and c^2. When you share these results what do you notice about columns 4 and 5? What does this tell you about all the right angled triangles that you have tested.



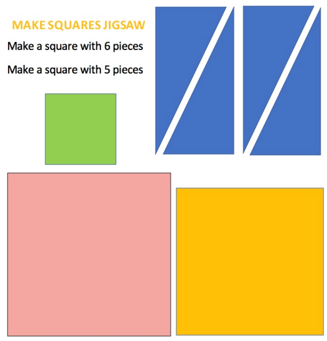
3. These 2 arrangements of the solutions are only 2 of 14 possible variations. There are 3 distinct solutions and 11 variations by reflection and rotation. Work as a group or class to make a poster showing all 14 possible variations and label them describing the transformations that give the variations.

INCLUSION AND HOME LEARNING GUIDE

THEME: Squares

Suggestions for Home Learning

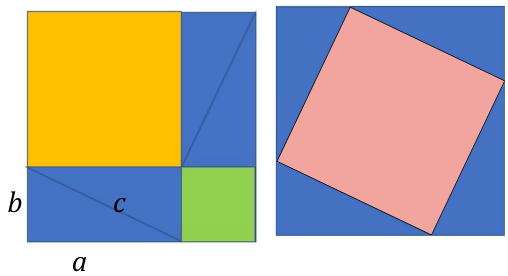
Young children



For young children the best activity is to play with the 7 pieces of the Make Squares Jigsaw with a frame into which either 5 pieces will fit or 6 pieces will fit.

See https://aiminghigh.aimssec.ac.za/make-squares-jigsaw/

Upper Primary



Start the session with the Make Squares Jigsaw. Give the learners the worksheet (page 8) and ask them to cut out the 7 pieces and to make a big square with 5 of the pieces and then another big square with 6 pieces. When they find solutions draw them on the board and ask the class what they notice about the arrangements of the pieces that fit together. Talk about the shapes and the areas of the pieces.

If they find solutions that are different from these two also draw them on the board. There are 3 different solutions and 11 variations that are reflections or rotations of one of the 3 solutions.

The learners could make their own jigsaw pieces. Each learner should **make 4 identical copies** of their right angled triangle, with their own different values of *a*, *b* and *c* for the edges. For example Mzu could call his a_M, b_M and c_M.

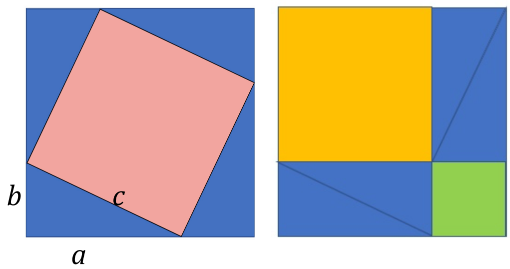
Two triangles together make a rectangle of area *ab* so the area of the triangle is $\frac{1}{2}ab$.

Square 5: edge length c area c^2

Square 6: edge length *b* area b^2

Square 7: edge length *a* area a^2

Lower Secondary



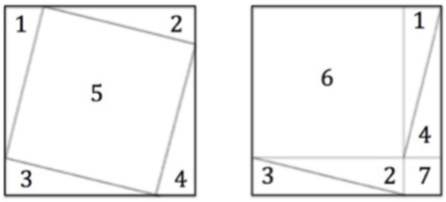
DO NOT TELL LEARNERS THAT THIS LESSON IS ABOUT PYTHAGORAS THEOREM.

Give the learners a piece of scrap paper and ask them to fold it into 4 so they can cut out 4 congruent right angled triangles from 4 layers of paper choosing their own shorter edge lengths, calling these lengths *a* and *b*.

The triangles must be right angled but can have any other angles. Then ask them to draw a square frame of edge length *a* + *b* and arrange the 4 triangles as shown in the first diagram.

Then they should arrange their 4 triangles in the frame as in the second arrangement and think about areas in the two arrangements.

Tell the learners to look at the two diagrams and make a list of what is the same in the arrangements of the pieces in the frame in the two ways and what is different. They should do this individually for a few minutes and then, if working in a group, compare their lists with someone else.



What do you notice?

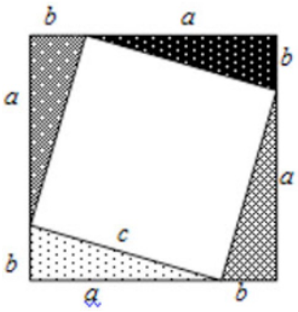
Ask the learners to draw diagrams like this using their own measurements *a* and *b*, and to label all the angles in the diagrams, and then to answer these questions.

Think about the areas in the diagrams.

- How can you be sure that the shapes labelled 5, 6 and 7 are squares?
- Apart from the four identical right angled triangles in each of the diagrams, what is left?
- What does this tell you about the area of square 5 compared with the areas of squares 6 and 7?

After the learners have worked on this activity discuss the necessary facts **building on what the learners have done for themselves**. Most learners will realise that the area of square 5 is equal to the sum of the areas of squares 6 and 7 and you can congratulate them on having themselves proved Pythagoras Theorem.

Years 9 and 10



Carry out the activity exactly as described for Lower Secondary. Ask the learners to draw their own diagrams and label them as in this diagram. Then ask them to write down and algebraic identity for the areas in the two diagrams. These identities are:

$(\boldsymbol{a}+\boldsymbol{b})^2 = 2\boldsymbol{ab}+\boldsymbol{c}^2$.

For the alternative arrangement the identity is

$(\boldsymbol{a}+\boldsymbol{b})^2 = 2\boldsymbol{ab}+\boldsymbol{a}^2+\boldsymbol{b}^2$.

This is a convincing demonstration of the fact that $\boldsymbol{a}^2+\boldsymbol{b}^2 = \boldsymbol{c}^2$.

It is also a beautiful example of connection between algebra and geometry because the area shows that multiplying $(\boldsymbol{a}+\boldsymbol{b})$ by $(\boldsymbol{a}+\boldsymbol{b})$ gives

$(\boldsymbol{a}+\boldsymbol{b})^2 = 2\boldsymbol{ab}+\boldsymbol{a}^2+\boldsymbol{b}^2 = \boldsymbol{a}^2+\boldsymbol{ab}+\boldsymbol{ab}+\boldsymbol{b}^2$

The results depend only on observation of the shapes and areas and how they fit together.

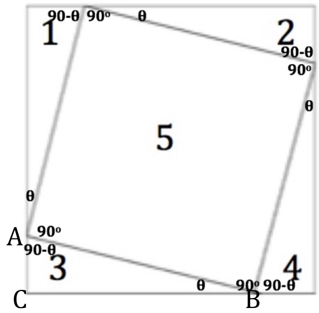
A formal proof requires a more formal argument.

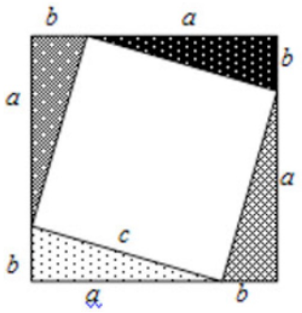
Proof

In quadrilateral 5 the edges are equal to *c* units because they are the hypotenuses of congruent right angled triangles. To prove it is a square we need to prove the angles are 90°.

At point A, two of the angles: $\boldsymbol{\theta}$ and $\mathbf{90 - \boldsymbol{\theta}}$, are the angles of the right angled triangles that must add up to 90°. The third angle at A (in quadrilateral 5) must be 90° because the three angles at A add up to 180° (angles on a straight line).

Similarly, the other 3 angles in 5 are right angles. This shape is a square of area c^2.





The outer frame is a square of edge length $\boldsymbol{a+b}$ and area $(\boldsymbol{a+b})^2$. This expression can be written as $(\boldsymbol{a+b})^2 = \boldsymbol{a^2 + b^2 + 2ab}$ (1)

The area inside the outer frame can be divided into 5 parts so the total area is: $\boldsymbol{c^2 + 4 \times \frac{1}{2}ab = c^2 + 2ab}$ (2).

From (1) and (2): $\boldsymbol{a^2 + b^2 + 2ab = c^2 + 2ab}$

and it follows that $\boldsymbol{a^2 + b^2 = c^2}$.

This completes the proof of Pythagoras Theorem.

Years 11, 12 and 13

If the students have not seen this proof before then they should carry out the experiment themselves and work through the proof as for Years 9 and 10.

They could also do the Make Squares Jigsaw and find the 14 variations of the solutions (see the NEXT section on page 2 and the Inclusion Guide for https://aiminghigh.aimssec.ac.za/make-squares-jigsaw/ .

Finding the 14 variations of solutions is a good problem solving activity and an excellent way to develop the skills of visualisation and working systematically and collaboratively. This activity helps students to develop a deeper understanding of Symmetry and Transformations which are of great importance in Higher Mathematics and in applications of mathematics.

If there is time students should do one or both of the activities:

Years 8 – 11 Riding on Pythagoras 1:

https://aiminghigh.aimssec.ac.za/years-8-11-riding-on-pythagoras-1/

Years 8 – 11 Riding on Pythagoras 2:

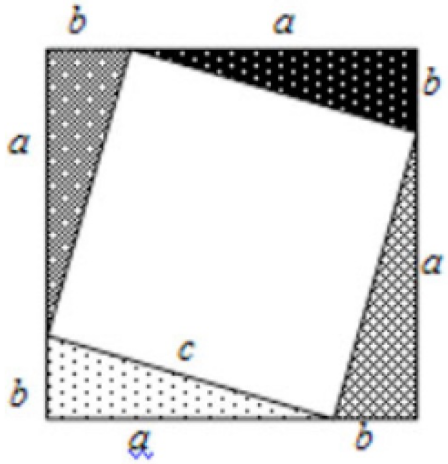
https://aiminghigh.aimssec.ac.za/years-8-11-riding-on-pythagoras-2/

Key questions

- Why is that angle a right angle (equal to 90°)?
- Can you label all the right angles in the diagrams?
- Can you label all the angles equal θ, that is to angle ABC?
- What do you know about the sum of the angles in a triangle?

- What do you know about the angles on a straight line?
- What are the lengths of the edges of shape 5? Can you explain why you know it is a square?
- What are the lengths of the edges of shape 6? Can you explain why you know it is a square?
- What are the lengths of the edges of shape 7? Can you explain why you know it is a square?

SOLUTION



The **4 congruent right angled triangles** numbered 1, 2, 3 and 4 in the diagrams all have edges of length *a, b* and c and angles 90°, θ and 90-θ.

The 4 congruent triangles are arranged in the squares as shown in the diagrams.

The same diagrams can be drawn for any lengths *a* and *b*.

In shape 5 all the angles are 90° because the angles on a straight line add up to 180°.

So shape 5 is a square with all the edges of length *c* and area c^2; shape 6 is a square with edges of length *a* and area a^2; and shape 7 is a square with edges of length *b* and area b^2.

In one arrangement the shapes 1, 2, 3 4, 6 and 7 fill the square frame of area $(a + b)^2$.

In the other arrangement the shapes 1, 2, 3 4 and 5 fill the square frame of area $(a + b)^2$.

The areas of shapes 6 and 7 in one arrangement and the area of shape 5 in the other arrangement take up the area: $(a + b)^2 - 2ab$ of the outer frame with 4 triangles removed. This proves that: the sum of the areas of shapes 6 and 7: $(a^2 + b^2)$ = area of shape 5 (c^2).

That is the sum of the areas of the squares on the two shorter edges of a right angled triangle is equal to the area of the square on the hypotenuse.

This proves Pythagoras Theorem for all right angled triangles because the same argument works with the same diagrams for any values of *a* and *b*.

Why do this activity?

A good teacher avoids whenever possible introducing a result to learners without showing them why it is true. It is bad practice to give one example (like the 3-4-5 triangle), or even several examples, and then to imply that the same result is true for all right angled triangles.

This activity can be used to lay the foundation for work on Pythagoras Theorem. It should be introduced BEFORE learners meet Pythagoras Theorem. It provides a visual proof of Pythagoras Theorem that is suitable for learners of different ages and attainment levels. It is a Low Entry Point High Ceiling learning activity.

This general proof of Pythagoras Theorem only requires a little simple reasoning and the knowledge that the angles of a triangle add up to 180°, that angles on a straight line add up to 180° and that the area of a square is equal to the square of the length of one edge.

In addition there is a lot of scope for work on transformations based on the different solutions that are produced by different learners. The class could make a wall display of the 14 variations naming each one with the name of the learner (or pair of learners) who first discovered that solution.

Learning objectives

In doing this activity students will have an opportunity to:

- learn or review the properties that define a square;
- for learners who have never heard of Pythagoras Theorem to experience thinking through a mathematical proof ***for themselves*** (guided re-invention) by using what they already know;
- develop a deeper understanding of symmetry, reflection and rotation.

Generic competences

In doing this activity students will have an opportunity to:

- **think mathematically,** reason logically and give explanations and proofs;
- **think flexibly**, be creative and innovative and apply knowledge and skills;
- **develop visualization** and the skill to interpret or create images to represent concepts;
- **work systematically** and collaboratively to find **all possible solutions** to a given problem.

Follow up

Make Squares Jigsaw (this would be good preparation before doing the Pythagoras Jigsaw activity): https://aiminghigh.aimssec.ac.za/make-squares-jigsaw/

Years 8 – 11

Riding on Pythagoras 1: https://aiminghigh.aimssec.ac.za/riding-on-pythagoras-1/

Riding on Pythagoras 2: https://aiminghigh.aimssec.ac.za/riding-on-pythagoras-2/

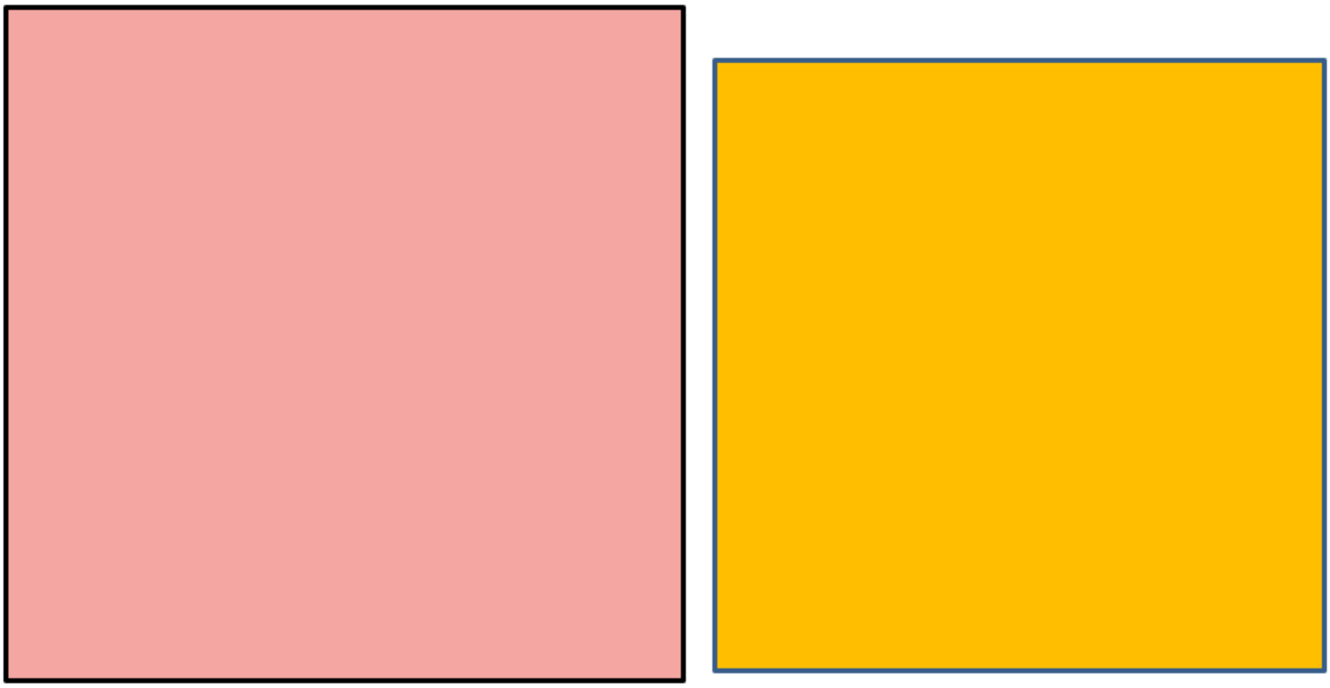
Go to the AIMSSEC YouTube channel https://www.youtube.com/mathstoys and subscribe to Maths Toys

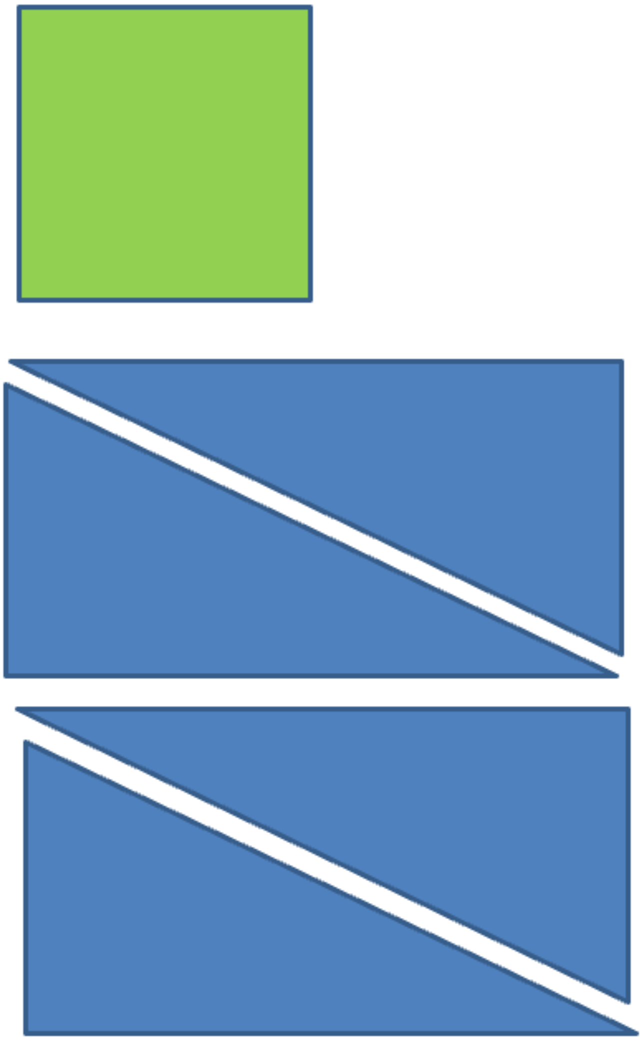
Go to the AIMSSEC AIMING HIGH website for lesson ideas, solutions and curriculum links: https://aiminghigh.aimssec.ac.za

Download the whole AIMSSEC collection of resources to use offline with the AIMSSEC App https://aimssec.app or download from Google Play ot App Store

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. New material will be added for Secondary 6.
For resources for teaching A level mathematics see https://nrich.maths.org/12339
The mathematics taught in Secondary 6 (East Africa) and in Europe is beyond the school curriculum for Grade 12 SA.

	Lower Primary Age 5 to 9	Upper Primary Age 9 to 11	Lower Secondary Age 11 to 15	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6





MAKE SQUARES JIGSAW

2-challenges in one!

Fit 5 of these pieces together to make a square

Fit 6 of these pieces together to make a square