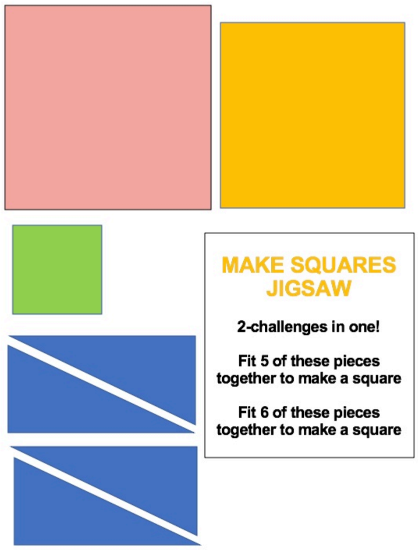
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)

AIMING HIGH

MAKE SQUARES JIGSAW Inclusion and Home Learning Guide is part of a double Learning Pack downloadable from the AIMING HIGH WEBSITE
https://aiminghigh.aimssec.ac.za/make-squares-jigsaw/ and
https://aiminghigh.aimssec.ac.za/pythagoras-jigsaw/
Together this pair of activities provides related activities on the Common Theme **SQUARES,** with guidance for home learning for all ages and learning stages from pre-school to school-leaving, together with resources for differentiation and inclusion in school lessons.
Choose what seems suitable for the age or attainment level of your learners.

MAKE SQUARES JIGSAW



This jigsaw has 7 pieces and 3 different solutions with variations of those solutions by reflection and rotation.

Can you find all the solutions?

Try the puzzle yourself.

Make a square with 5 pieces, then make a square with 6 pieces.

The instructions for making your own jigsaw pieces are on page 3.

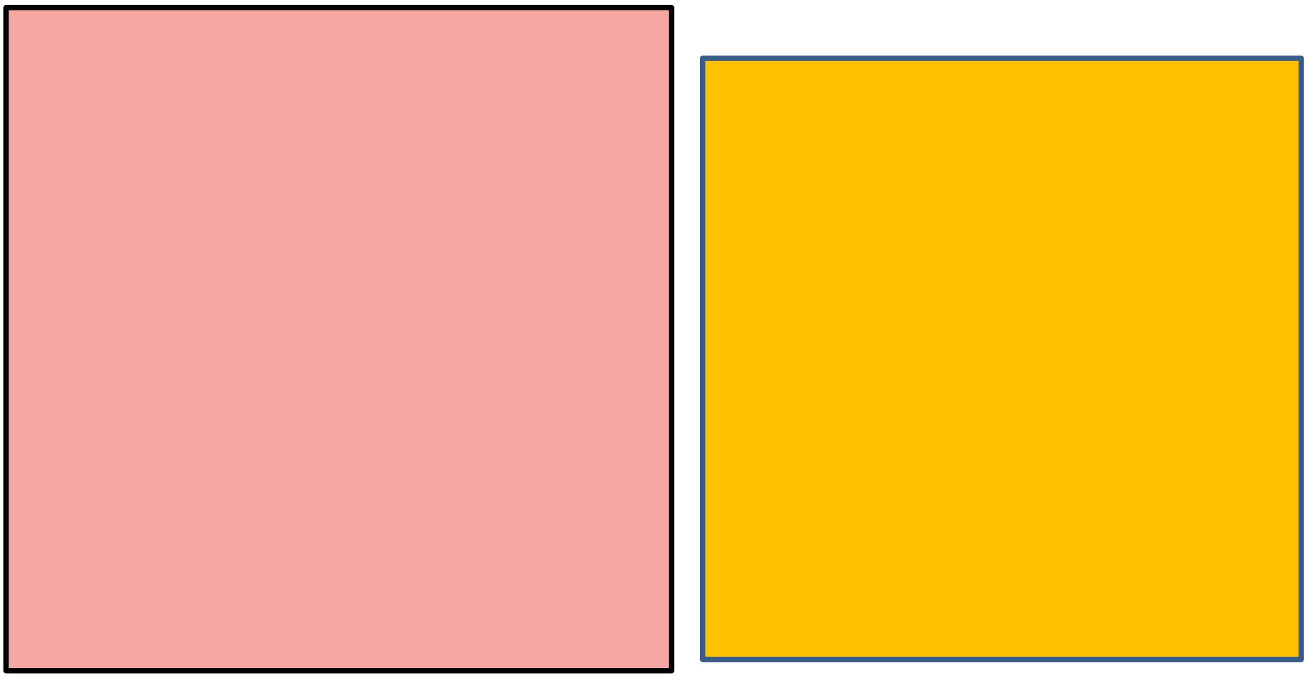
HELP

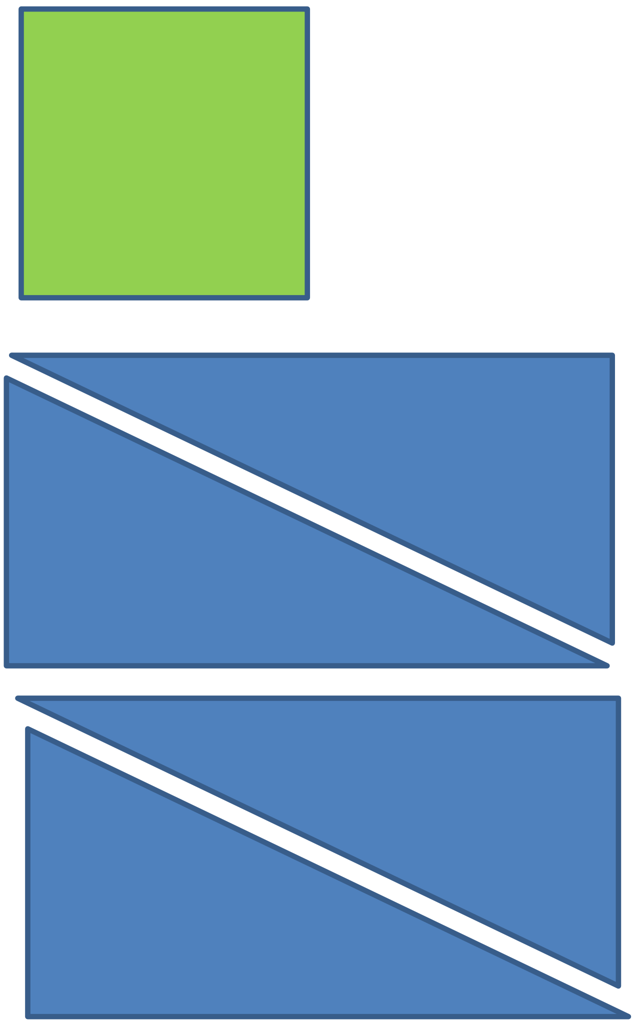
It might help to investigate the lengths of all the edges and use what you know about these lengths to help you to solve the puzzle.

NEXT

1. Investigate the areas of the pieces and the areas of the squares in the solution.

2. Investigate different variations of the solutions by observing how the 4 triangles are positioned and transformations of the solutions that produce variations of that solution.





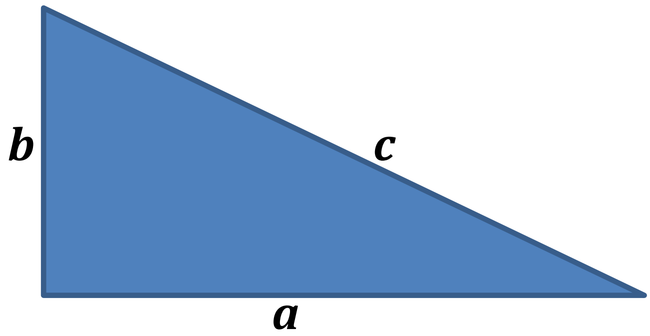
MAKE SQUARES JIGSAW

2-challenges in one!

Fit 5 of these pieces together to make a square

Fit 6 of these pieces together to make a square

MAKE YOUR OWN 'MAKE SQUARES JIGSAW' PIECES



You can use scrap paper or card but you need it to have right angles at the corners.

Start with **any right angled triangle**. Measure the lengths of the edges and call these *a*, *b* and *c* units.

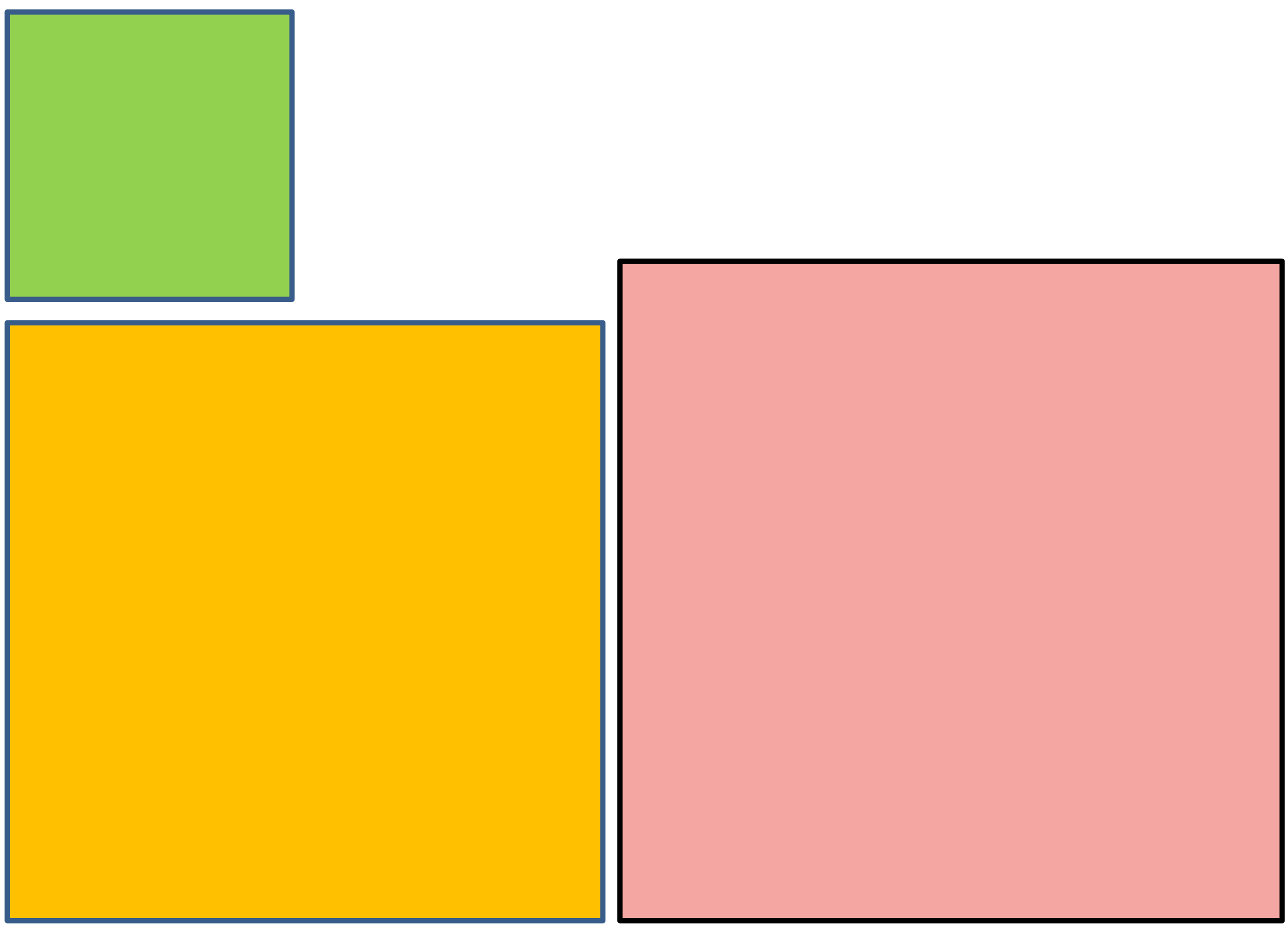
You need to make 4 triangles that are all identical and 3 squares, one to match each edge of the triangle: $a \times a$, $b \times b$ and $c \times c$.

Use the corners of a piece of paper to give you right angles.

A pin prick to mark a point before drawing the line helps to make the drawing more accurate.

Use the first triangle that you make to give you the lengths of the edges for the other triangles and for the edges of the 3 squares.

You don't need to do any measuring and so you only need a ruler to draw straight lines.



HOME LEARNING AND INCLUSION GUIDE

THEME: Squares

Suggestions for Teaching or Home Learning

Young children

The jigsaw is suitable for young children especially if they fit the pieces into a frame and they are shown one solution, and then the pieces are jumbled and they have to reassemble them.

Upper Primary

Learners can cut out the pieces to make the puzzles. When they have found solutions they should draw diagrams in their workbooks. Encourage learners to find variations of the solutions working as a class and sharing ideas. They can come back to it another day if they don't find solutions the first day.

Lower Secondary

Learners should find solutions and variations working in pairs and sharing ideas as a class. They should draw diagrams to record the different solutions and their variations and write a list of what they notice. They should describe any symmetries that they see. Some learners may make observations about areas and be able to discover Pythagoras Theorem.

Years 9 and 10

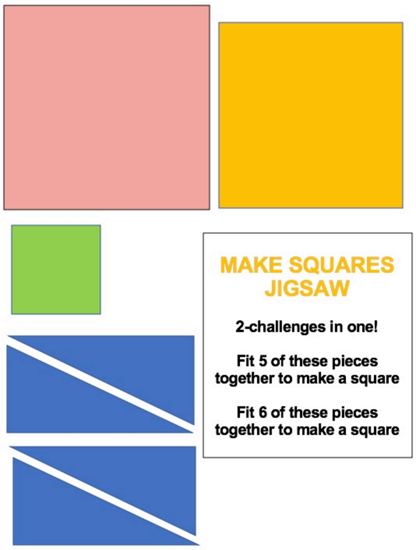
This activity should be done BEFORE learners ever meet Pythagoras Theorem. First solve the puzzle. Study the solutions. Describe and record the variations of the solutions and the symmetries involved. Investigate the areas of all the pieces and the areas of the square outlines (frame) into which the solutions fit. Learners should be able to notice from the solutions to this jigsaw that the area of the largest square (with edge matching the hypotenuse of the triangle) is equal to the sum of the areas of the two smaller triangles. In this way learners actually discover Pythagoras Theorem for themselves.

Years 11, 12 and 13

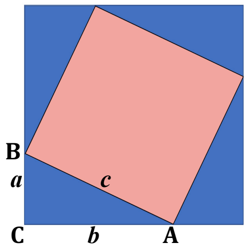
The challenge is to use the solutions of this puzzle to construct a formal proof of Pythagoras Theorem.

Extension There are 56 variations of the solutions and they can be classified into 9 distinct sets, one set containing 2 variations, two sets containing 4 variations and eight sets containing 8 variations. In each set the variations are all reflections or rotations of each other. Find as many of the variations as you can and classify them into sets. In each set choose one representative and describe the transformations of your chosen representative that give the other elements of the set.

Key Questions

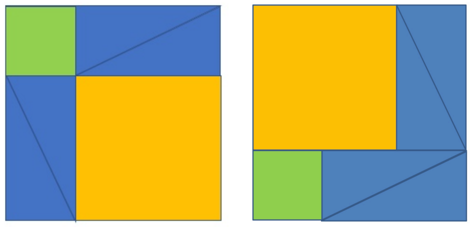


Solution 1



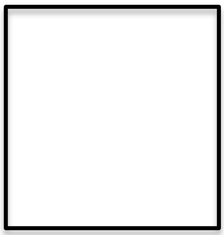
5-piece solution

Solution 2 **Solution 3**



2 different 6-piece solutions

The solutions fit exactly into the same square frame.



You can make the four congruent right-angled triangles any shape and size but they must be identical. The squares must have edge lengths equal to the edges of the triangle.

Based on the fact that the triangles are right angled, answer the following questions:

1. Draw your own diagrams for a 5-piece and a 6-piece solution and label all the right angles in the diagrams.
2. How do you know for certain that the frame is square?
3. Label all the angles equal to angle ABC, calling this angle θ.
4. Label the other angles and lengths in the diagrams.
5. Using the angles that you have labelled explain how you can be sure that, if the pieces were cut out accurately, they would fit exactly into the square frame.

Put the 4 triangles in position in the frame for solution 1

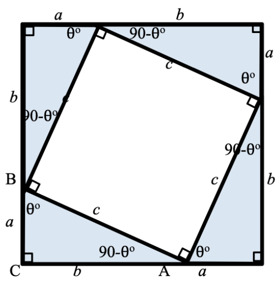
6. What can you say about the space left uncovered?
7. What is the area of the uncovered space?

Move the 4 triangles into position for solution 2

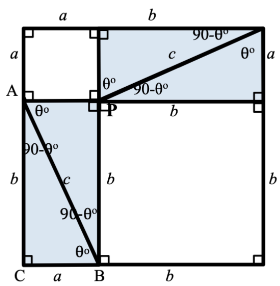
8. What can you say about the space left uncovered?
9. What is the area of the uncovered space?
10. What do you notice about the areas of the squares in these 2 solutions?
11. What are the lengths of the edges of the frame in terms of *a* and *b*?
12. What is the area of the frame?
13. Use the areas of the pieces in solution 2 to prove that $(a + b)^2 = a^2 + b^2 + 2ab$.

Solutions

1. *Draw your own diagrams for solutions 1 and 2 and label all the right angles in the diagrams*



Solution 1



Solution 2

1. *How do you know for certain that the frame is square?*

   The right angles in the 4 congruent copies of ΔABC fit into the corners of the outer frame so the outer frame is a square with edge length $a + b$.
2. *Label all the angles equal to angle ABC, calling this angle* θ.
3. *Label the other angles and lengths in the diagrams.*
4. *Using the angles that you have labelled explain how you can be sure that, if the pieces were cut out accurately, they would fit exactly into the square frame.*

In Solution 1:
We want to show that the quadrilateral that appears to be a tilted square actually is a square. The angles in the two triangles at each vertex of the quadrilateral add up to 90^o (because $\theta^o + (90-\theta)^o = 90^o$). So the 4 angles in the quadrilateral are each 90^o (because the angles on a straight line add up to 180^o) showing that this quadrilateral is a square with edge length c. Therefore the largest square puzzle piece fits exactly into that space.

In Solution 2:
The outer frame is a square with edge length $a + b$. Each pair of triangle forms a rectangle of edge lengths a and b which fit into the corners of the frame with four vertices of the triangles meeting at an internal point labelled P in the diagram. The two smaller square puzzle pieces measuring a by a and b by b fit exactly into the remaining spaces.

Put the 4 triangles in position in the frame for solution 1

6. *What can you say about the space left uncovered?*

The space left uncovered is a square with edge length c by c.

7. *What is the area of the uncovered space?*

The area of the space left uncovered is c^2 square units.

Move the 4 triangles into position for solution 2

8. *What can you say about the space left uncovered?*

Checking that the angles at point P add up to 360^o shows that the remaining spaces left uncovered in the frame are squares with edge lengths a by a and b by b.

9. *What is the area of the uncovered space?*

The area of the space left uncovered is a^2+b^2 square units

10. *What do you notice about the areas of the squares in these 2 solutions?*

The areas of the spaces left uncovered in the 2 solutions must be equal because it is the total space less the areas of the 4 triangles. So c^2 (from Solution 1) is equal to a^2+b^2 (from Solution 2) that is $c^2 = a^2+b^2$, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two edges of the right angles triangle ΔABC.

This proves Pythagoras's Theorem

11. *What are the lengths of the edges of the frame in terms of a and b?*

$a + b$

12. *What is the area of the frame?*

$(a + b)^2$

13. *Use the areas of the pieces in solution 2 to prove that* $(a + b)^2 = a^2 + b^2 + 2ab$.

In Solution 2 the 4 triangles make up 2 rectangles each of area ab and the two smaller squares have areas a^2 and b^2 so the total area of the pieces in Solution 2 is $a^2 + b^2 + 2ab$.

Using the result in 12. Above, and equating the two expressions for the area inside the frame, gives $(a + b)^2 = a^2 + b^2 + 2ab$. This is a geometric result illustrating, equivalent to, but not dependent on, the algebraic rule for multiplying two binomial expressions.

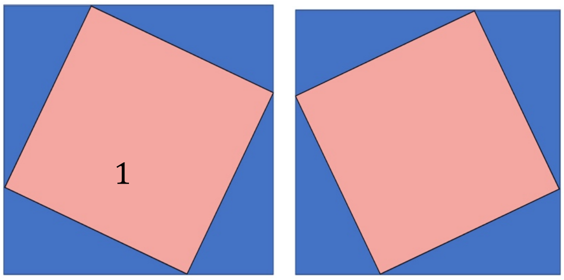
EXTENSION OF THE MAKE SQUARES PUZZLE. This extension activity explores all the variations of the solutions to the puzzle. The solutions can be classified into 9 sets as shown below. There are 2 variations of the 5-piece solution and 56 variations of the 6-piece solution. The variations are rotations or reflections of one of the 9 solutions.

Different learners will find different variations and they could identify their own solutions from the diagrams below. Between them the class might try to find as many variations as possible although it is unlikely that they will find all possibilities unless a lot of time is spent on it. As a 'high ceiling challenge' some learners might identify the sets and describe the transformations.

Key Questions to draw attention to the different possible variations in the solutions.

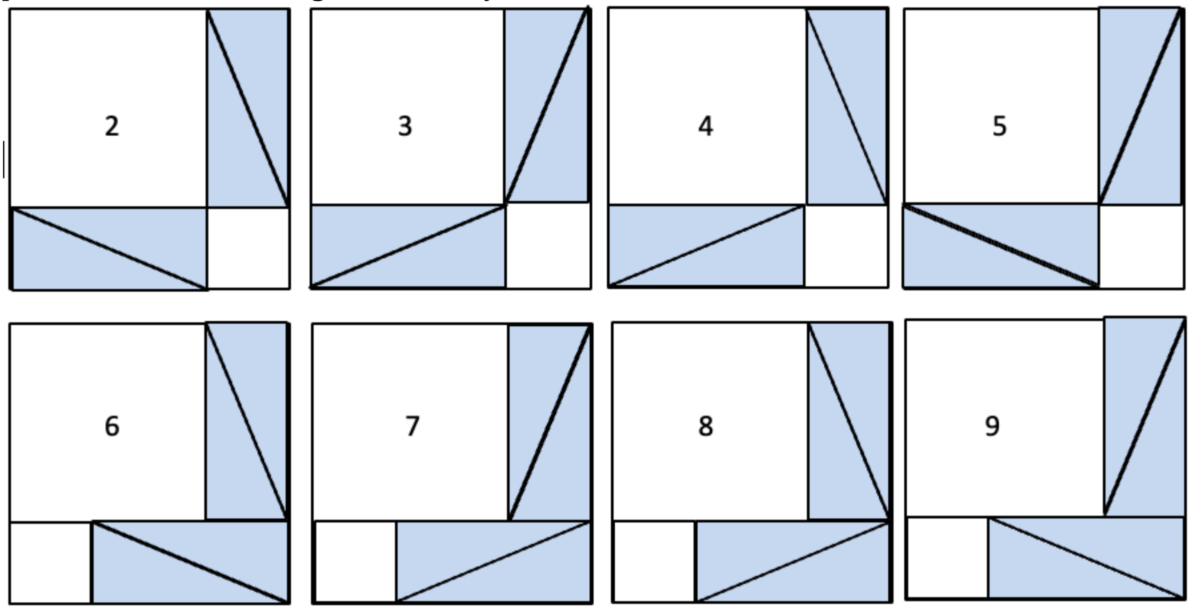
- What do you notice about the positions of the triangles in that solution?
- Can you see any rotational symmetry in that solution?
- Are there other variations of the solutions that can be produced by reflection or rotation?

SOLUTION



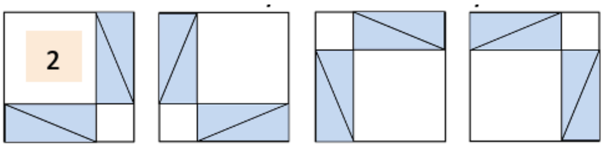
Solution 1 is the 5-piece solution with rotational symmetry of order 4. It has 2 variations: the image of one of the variations by reflection either vertically or horizontally gives the other variation.

Solutions 2, 3, 4 and 5 are distinct solutions although they all look alike until you look at the positions of the 4 triangles. Similarly, 6, 7, 8 and 9 are all distinct.

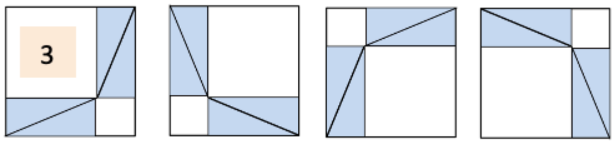


Solutions 2 and 3 are symmetric and each has 4 variations. By rotating and reflecting the solution labelled 4 we get altogether 8 variations of this solution. Similarly, solutions 5, 6, 7, 8 and 9 each have 8 variations. Altogether there are 2 variations of the 5-piece solution and 56 variations of the 6-piece solution.

For each set of variations, we could choose any element of the set and produce the same set of variations by reflecting and rotating the chosen one, but the variations would appear in a different order.



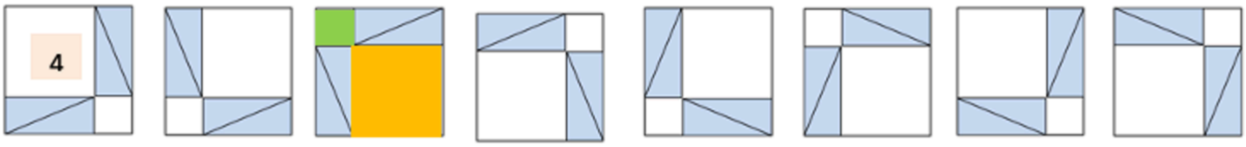
Solution 2 has 4 variations	2 rotated by 90° clockwise or reflected in a vertical mirror	2 rotated by 180° or reflected in a diagonal	2 rotated by 90° anti-clockwise in a horizontal mirror



Solution 4 has 4 variations	4 rotated by 90° clockwise or reflected in a vertical mirror	4 rotated by 180° or reflected in a diagonal	4 rotated by 90° anti-clockwise in a horizontal mirror

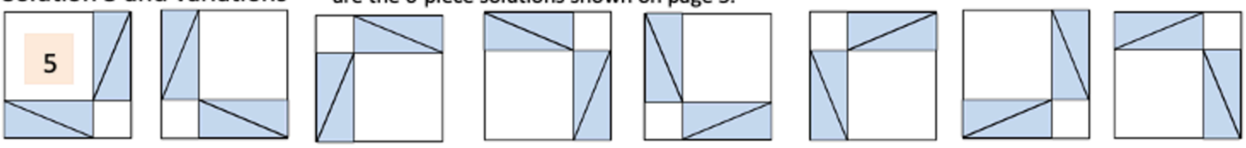
Solutions 2 and 3 are symmetric and they each have 4 variations or repeats

Solution 4 has 8 variations	4 rotated by 90° clockwise	4 rotated by 180°	4 rotated by 90°anti-clockwise	4 reflected in a vertical mirror	4 reflected in a diagonal mirror	4 reflected in the other diagonal	4 reflected in a horizontal mirror

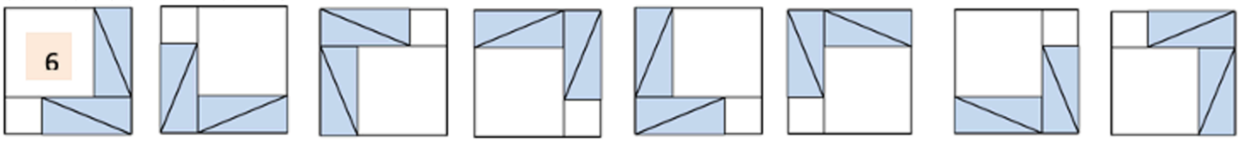


The variations with the orange and green squares are the 6-piece solutions shown on page 5.

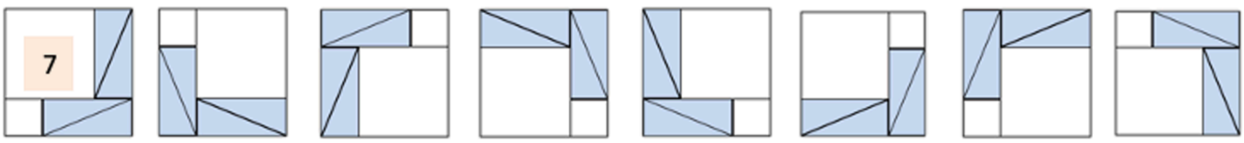
Solution 5 and variations



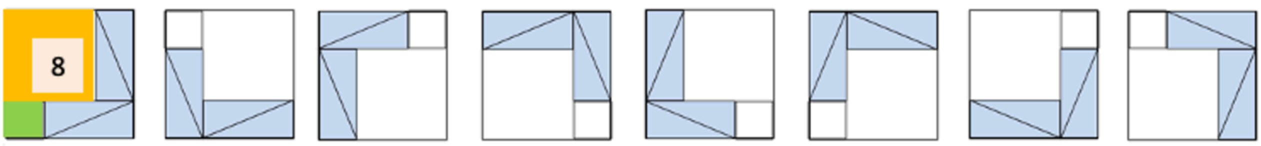
Solution 6 and variations



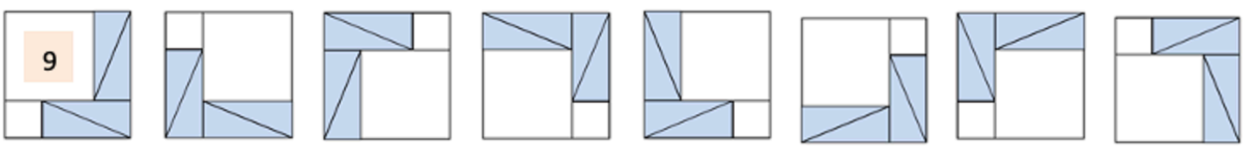
Solution 7 and variations



Solution 8 and variations



Solution 9 and variations



Why do this activity?

This activity can be used to lay the foundation for work on Pythagoras Theorem. It should be introduced BEFORE learners meet Pythagoras Theorem. It provides a visual proof of Pythagoras Theorem that is suitable for learners of different ages and attainment levels. It is a Low Entry Point High Ceiling learning activity.

The 5-piece and 6-piece solutions fit into the same frame and it is easy to see that the spaces not covered by the triangles must be the same area so that the area of the largest square (the square on the hypotenuse) is the sum of the areas of the two smaller squares (the squares on the other two edges of the triangle). The beauty of this informal proof of Pythagoras Theorem is that it works for any right angles triangle and this will be apparent to the class if the learners make their own puzzle pieces. The puzzle can be solved by younger learners without reference to Pythagoras Theorem and this informal proof is quite sufficient as an introduction to Pythagoras Theorem.

A good teacher avoids whenever possible introducing a result to learners without showing them why it is true. It is bad practice to give one example (like the 3-4-5 triangle), or even several examples, and then to imply that the same result is true for all right angled triangles.

A formal proof of Pythagoras Theorem only requires a little simple reasoning and the knowledge that the angles of a triangle add up to 180^o, that angles on a straight line add up to 180^o and that the area of a square is equal to the square of the length of one edge.

In addition there is a lot of scope for work on transformations based on the different solutions that are produced by different learners. The class could make a wall display of the variations naming each one with the name of the learner (or pair of learners) who first discovered that solution.

Learning objectives

In doing this activity students will have an opportunity to:

- learn or review the properties that define a square;
- for learners who have never heard of Pythagoras Theorem to engage in thinking mathematically about triangles, squares and their areas and discover a proof of Pythagoras Theorem for themselves;
- develop a deeper understanding of symmetry, reflection and rotation.

Generic competences

In doing this activity students will have an opportunity to:

- **work systematically** and collaboratively to find **all possible solutions** to a given problem.
- **develop visualization** skills.

Follow up

There is an extension of this activity in
Pythagoras Jigsaw https://aiminghigh.aimssec.ac.za/pythagoras-jigsaw/
See also: **Square It Game** https://aiminghigh.aimssec.ac.za/square-it-game/
How Many Squares https://aiminghigh.aimssec.ac.za/how-many-squares/