SANDWICHES

112233

Suppose you have two 1's, two 2's and two 3's.
Arrange these six digits in a list so that:

between the two 1's there is one digit giving 1?1,
between the two 2's there are two digits giving 2??2,
between the two 3's there are three digits giving 3???3.

Can you do the same if you only have 1's and 2's? Explain your answer.

Can you do the same if you include two four's, and between the two 4's there are four digits?

The illustration shows a 7-sandwich using the pairs of all the whole numbers from 1 to 7.

71316435724625

Can you make another 7 sandwich?

For some values of n it is possible to make n-sandwiches using pairs of every positive integer from 1 to n, and arranging the list according to the rules given above, and for other values of n it is not possible. The big challenge is to find when it is and when it is not possible.

HELP

You may want to stop with n=4.

NEXT

The process of working systematically through all possible cases can be tedious and time consuming. This is where discussion of the best systematic approach, and sharing the work of searching for solutions and checking, offers a valuable learning experience. Such work prepares learners for using computers in problem solving, not just in mathematics but also in other fields.

In order to construct a computer program you have first to plan a systematic approach to the problem. Another extension could be to write a computer program to find solutions and to test which values of n yield n-sandwiches.

The alternate colours are the key to the proof that for some values of n you can make n-sandwiches and for some values of n it is impossible.

7 1 3 1 6 4 3 5 7 2 4 6 2 5

Finally, there is a method of proving that for certain values of n it is impossible to make n-sandwiches. The argument is simple and requires only very elementary mathematics. The proof falls into the 'Aha' category - once you see it, it seems obvious and amazingly simple, but the choice of method called for real inspiration in the first place. The picture gives a clue, can you think of a proof?

7-sandwiches

1)	1 7 1 2 5 6 2 3 4 7 5 3 6 4
2)	1 7 1 2 6 4 2 5 3 7 4 6 3 5
3)	1 6 1 7 2 4 5 2 6 3 4 7 5 3
4)	1 5 1 6 7 2 4 5 2 3 6 4 7 3
5)	1 4 1 5 6 7 4 2 3 5 2 6 3 7
6)	1 4 1 6 7 3 4 5 2 3 6 2 7 5
7)	1 6 1 3 5 7 4 3 6 2 5 4 2 7
8)	1 5 1 7 3 4 6 5 3 2 4 7 2 6
9)	1 5 1 6 3 7 4 5 3 2 6 4 2 7
10)	1 5 1 4 6 7 3 5 4 2 3 6 2 7
11)	5 1 7 1 6 2 5 4 2 3 7 6 4 3
12)	4 1 7 1 6 4 2 5 3 2 7 6 3 5
13)	4 1 6 1 7 4 3 5 2 6 3 2 7 5
14)	7 1 3 1 6 4 3 5 7 2 4 6 2 5
15)	7 1 4 1 6 3 5 4 7 3 2 6 5 2
16)	6 1 5 1 7 3 4 6 5 3 2 4 7 2
17)	4 6 1 7 1 4 5 2 6 3 2 7 5 3
18)	7 3 1 6 1 3 4 5 7 2 6 4 2 5
19)	4 6 1 7 1 4 3 5 6 2 3 7 2 5
20)	5 6 1 7 1 3 5 4 6 3 2 7 4 2
21)	7 4 1 5 1 6 4 3 7 5 2 3 6 2
22)	5 7 1 4 1 6 5 3 4 7 2 3 6 2
23)	3 6 7 1 3 1 4 5 6 2 7 4 2 5
24)	5 7 4 1 6 1 5 4 3 7 2 6 3 2
25)	2 6 7 2 1 5 1 4 6 3 7 5 4 3
26)	4 5 6 7 1 4 1 5 3 6 2 7 3 2

There are mirror images of each of these sol

NOTES FOR TEACHERS

SOLUTION

3-sandwiches: 312132 and its mirror image 231213

You cannot make 2-sandwiches with only ones and twos, because between the two twos there must be two digits, which have to be ones as these are the only available digits, but that means that between the two ones there are no digits, so this is not possible.

4-sandwiches: The only 4-sandwiches are 41312432 (and its mirror image).

7-sandwiches: There are altogether 26 (and their mirror images).

There are no 5-sandwiches and no 6-sandwiches.

8-sandwiches: There are altogether 150 of these (and their mirror images).

See the **Follow Up** on pages 4 and 5 for a simple proof that n-sandwiches can only be made when n=4m or n=4m-1 (for integers n and m). For example solutions do not exist for n=5 or n=6 but solutions do exist for n=7 (4×2 -1) and n = 8 (4×2).

There are no n-sandwiches for n=1, 2, 5, 6 or 9 or any number that leaves a remainder of 1 or 2 when divided by 4.

Why do this activity?

The Sandwiches Problem offers a challenge for everyone at all levels. It is valuable in primary schools because it gives young learners an opportunity to work out a proof and to explain it. They can find the solution for the 3-sandwich (consisting of 1's, 2's and 3's) and discover that it is impossible for the 2-sandwich. Even very young learners can explain why (prove that) it is impossible to find a solution for the 2-sandwich. You may decide to stop at 3-sandwiches or try to find 4-sandwiches as well.

At upper primary and lower secondary level, as there are many solutions in the case of 7-sandwiches and 8-sandwiches, the problem provides an opportunity for many individual learners to have success in discovering their very own solution, different to any that have already been found.

Having investigated 2, 3, 4 and 7-sandwiches, if learners in your class have been encouraged to ask "What if..." and look for generalisations, the natural question is "What about 5-sandwiches and 6 sandwiches?"

The experience of learning to think mathematically offered by this problem is equally valuable to learners in upper secondary school.

Learning objectives

In doing this activity students will have an opportunity to:

- practise pattern recognition and be able to search systematically for arrangements of sets of numbers that obey the rules for the pattern.
- give a reasoned explanation of why it is impossible to make the pattern for n=2

Generic competences

In doing this activity students will have an opportunity to:

- **think mathematically,** reason logically and give explanations and proofs;
- **work in a team using a systematic method** so as to share the workload:
- **communicate** in writing, speaking and listening:
    - exchange ideas, criticise, and present information and ideas to others
    - analyze, reason and record ideas effectively.

Suggestions for teaching

It is helpful, particularly for young learners, to have digits to rearrange (either plastic or simply written on paper or card).

It is a good idea to have a 'Challenge-Chart' on the classroom wall where new solutions can be written up as people discover them.

As there are altogether 26 solutions for n=7, this problem calls for you to work systematically in order to find them all.

Key questions

- What sandwiches can you make?
- Can you make 2-sandwiches and if not why not?
- Are any sandwiches the same looked at in different ways?
- Is it possible to make 5-sandwiches?
- For which values of n can n-sandwiches be made and for which values of n is it impossible?
- Why?

Follow up

Study the proof below and try to explain it to someone else.

Impossible Sandwiches

Article by Adam Huby and Paul Cockayne

This delightful, simple and completely general proof about when solutions exist and when they do not exist was contributed by Alan Parr from his games magazine 'Hopscotch'.

The Sandwiches problem is adapted from the NRICH task of the same name published in September 1997, with permission of the University of Cambridge. All rights reserved. You may like to study the other solutions in the NRICH Archive contributed by school students.

Here is the puzzle again:

Suppose you have two 1s, two 2s and two 3s. Arrange these six digits in a list so that:

- between the two 1s there is one digit giving 1?1,
- between the two 2s there are two digits giving 2??2,
- and between the two 3s there are three digits giving 3???3.

Can you do the same if you only have 1s and 2s? Explain your answer.

Can you do the same if you include two fours, and between the two 4s there are four digits?
Here is a solution using 5s, 6s and 7s as well: 71316435724625. Find other solutions with all these digits.

Here is Adam Huby and Paul Cockayne's stunningly simple proof that solutions only exist for n=4m or n=4m-1.... To make the terminology a bit simpler, colour digits in the "solution number" or "sandwich", alternately red and blue:.

7 1 3 1 6 4 3 5 7 2 4 6 2 5

Then all odd numbers in the solution number will be either both red or both blue. Even numbers will always be one red, one blue. Since the final solution contains an equal number of red and blue digits, the problem is only soluble if we have an even number of odd numbers.
The only n-sandwiches which have an even number of odd numbers are those where n is a multiple of 4 or 1 less than a multiple of 4. (For example 9-sandwiches don't exist because if they did they would contain 1s, 3s, 5s, 7s and 9s)
Obvious, isn't it?
Finally, just to prove that he had done it, Adam Huby gave the following solution for n = 67.
67 65 66 62 60 64 57 63 54 52 61 49 47 59 44 58 41 39 56 36 55 33 30 53 26 24 51 20 50 5 9 48 10 12 46 5 45 7 3 43 9 42 3 10 40 7 12 38 20 37 24 26 35 30 34 33 36 39 41 44 47 49 52 54 57 60 62 65 67 66 64 63 61 59 58 56 55 53 51 50 48 46 45 43 42 40 38 37 35 34 32 29 31 28 25 23 21 27 18 6 13 11 8 22 4 2 6 19 2 4 17 8 16 11 13 14 15 18 21 23 25 29 28 32 31 27 22 19 17 16 14 1 15 1

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. New material will be added for Secondary 6.
For resources for teaching A level mathematics see https://nrich.maths.org/12339
Note: The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is beyond the school curriculum for Grade 12 SA.

	Lower Primary or Foundation Phase Age 5 to 9	Upper Primary Age 9 to 11	Lower Secondary Age 11 to 14	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6