AIMSSEC

AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

QUADRATIC SEQUENCES

1. A sequence is defined by the formula $a_n = n^2 - 2n + 1$.

(a) Calculate the first 8 terms of the sequence.

(b) Calculate the first differences between the terms.

(c) Calculate the second differences between the terms.

(d) What can you say about the results you obtained?

HELP

The word QUADRATIC refers to terms of the second degree (or squared). In Algebra, we use the quadratic formula to solve second degree equations.
A sequence which is quadratic in nature will always have the n^{th} term in the form:

$T_n = an^2 + bn + c$ where a, b and c are constants.

When:	$n = 1$		$n = 2$		$n = 3$		$n = 4$		$n = 5$
T_n:	$a + b + c$		$4a + 2b + c$		$9a + 3b + c$		$16a + 4b + c$		$25a + 5b + c$
1st difference →		$3a + b$		$5a + b$		$7a + b$		$9a + b$	
2nd difference →			$2a$		$2a$		$2a$		$2a$

Fig. 1

Try solving the following problems:

1. What is the n^{th} term of the following sequence?
   2 5 10 17 26 37 …

Determine whether the sequence is quadratic. Find the first differences, the second differences. What do you notice?

The n^{th} term should be of the form $T_n = an^2 + bn + c$.

Find expressions for T_1, T_2, and T_3. Form equations using these expressions and the 1st, 2nd, and 3rd terms, **respectively**, of the given sequence.

Form equations from: $T_2 - T_1$ and $T_3 - T_2$.
Solve the two equations simultaneously to obtain values of a, b, and c.

This should lead you to the solution $T_n = n^2 + 1$. Notice that the value of $b = 0$ makes the middle term (bn) in the expression $an^2 + bn + c$ disappears, hence $T_n = n^2 + 1$.

2. What is the n^{th} term of the following sequence?
   6 11 18 27 38 51…

Determine the 1^{st} and 2^{nd} differences of the sequence.
Is this a linear or quadratic sequence?

From Fig. 1 above and your first and second differences of the sequence: 6, 11, 18, 27, 38, 51, … we can establish the following relations:

(i) $2a = 2$ (2^{nd} difference) ………………………1

(ii) $3a + b = 5$ ($T_2 - T_1$) …………………………2

(iii) $a + b + c = 6$ (T_1) ……………………………3

This is true for any quadratic sequence

Solve 1, 2, and 3 simultaneously to obtain $T_n = n^2 + 2n + 3$

3. Find the n^{th} term of the following sequence:
   0, 6, 20, 42, 72, ….

   Obtain the 1^{st} and 2^{nd} differences.
   What do you notice?
   From your results, demonstrate that this leads to three equations, namely:

   $2a = 8$ ……………………………………………1
   $3a + b = 6$ ………………………………………..2
   $a + b + c = 0$ ……………………………………...3

   Solve 1, 2, and 3 simultaneously to obtain $T_n = 4n^2 + - 6n + 2$

NEXT

1.
(a) Calculate the first 6 terms of the sequence generated by the quadratic formula:
$a_n = 3n^2 - n - 2$

n = 1	$a_1 =$
n = 2	$a_2 =$
n = 3	$a_3 =$
n = 4	$a_4 =$
n = 5	$a_5 =$
n = 6	$a_6 =$

∴ The 6 terms of the sequence are: ………………………………………………………………………

(b) Show that the second difference of this sequence is a constant equal to 6.

(c) Plot a graph of T_n *versus* n, where n is an integer. The ‘horizontal’ axis gives the number, n, of the term, the ‘vertical’ axis the value, a_n, of the term.

SOLUTION

The first term, which we shall call a_1, is formed by substituting 1 for n in the formula:

$a_1 = 1^2 – 2(1) + 1 = 1 – 2 +1 = 2 – 2 = 0$

Similarly, the second term will be given by:

$a_2 = 2^2 - 2(2) + 1 = 4 – 4 + 1 = 5 – 4 = 1$

The third term will be:

$a_3 = 3^2 – 2(3) + 1 = 9 – 6 + 1 = 10 – 6 = 4$

The fourth term will be:

$a_4 = 4^2 – 2(4) + 1= 16 – 8 + 1 = 17 – 8 = 9$

The fifth term will be:

$a_5 = 5^2 – 2(5) + 1 = 25 – 10 + 1 = 26 – 10 = 16$

The sixth term will be:

$a_6 = 6^2 – 2(6) + 1 = 36 – 12 + 1 = 37 – 12 = 25$

The seventh term will be:

$a_7 = 7^2 – 2(7) + 1 = 49 – 14 + 1 = 50 – 14 = 36$

The eighth term will be:

$a_8 = 8^2 – 2(8) + 1 = 64 – 16 + 1 = 65 – 16 = 49$

Therefore, the sequence is:

0, 1, 4, 9, 16, 25, 36, 49, …

(b) 0 1 4 9 16 25 36 49

First differences 1 3 5 7 9 11 13

(c) 0 1 4 9 16 25 36 49

First differences 1 3 5 7 9 11 13 — 49 - 36

Second differences 2 2 2 2 2 2 — Constant

(d) The differences between the first differences are constant. They are all equal to 2 in this case. These are called the second differences, as shown above. From these results one can conclude that a sequence of numbers has a quadratic pattern when its sequence of second differences is constant.

HELP

1. 2 5 10 17 26 37…

3 5 7 9 11

2 2 2 2

The sequence is quadratic as the second difference leads to a constant, 2 in this case.

$$\left.\begin{array}{l} 3a + b = 3 \ldots (1) \\ 5a + b = 5 \ldots (2) \end{array}\right\}$$ Solving (1) and (2) simultaneously gives: a = 1, b = 0.

From: a + b + c = 2 $\xrightarrow{yields}$ $c = 1$ $\therefore$ $T_n = n^2 + 1$. Notice that b = 0.

In the same vein, 2 and 3 can be solved.

NEXT

1. (a) First 6 terms of $a_n = 3n^2 - n - 2$:

n = 1	$a_1 = 3\,(1^2) - 1 - 2 = 3 - 1 - 2 = 0$
n = 2	$a_2 = 3\,(2^2) - 2 - 2 = 12 - 2 - 2 = 8$
n = 3	$a_3 = 3\,(3^2) - 3 - 2 = 9 - 3 - 2 = 22$
n = 4	$a_4 = 3\,(4^2) - 4 - 2 = 12 - 4 - 2 = 42$
n = 5	$a_5 = 3\,(5^2) - 5 - 2 = 15 - 5 - 2 = 68$
n = 6	$a_6 = 3\,(6^2) - 6 - 2 = 18 - 6 - 2 = 100$

$\therefore$ the 6 terms of the sequence are: 0 8 22 42 68 100.

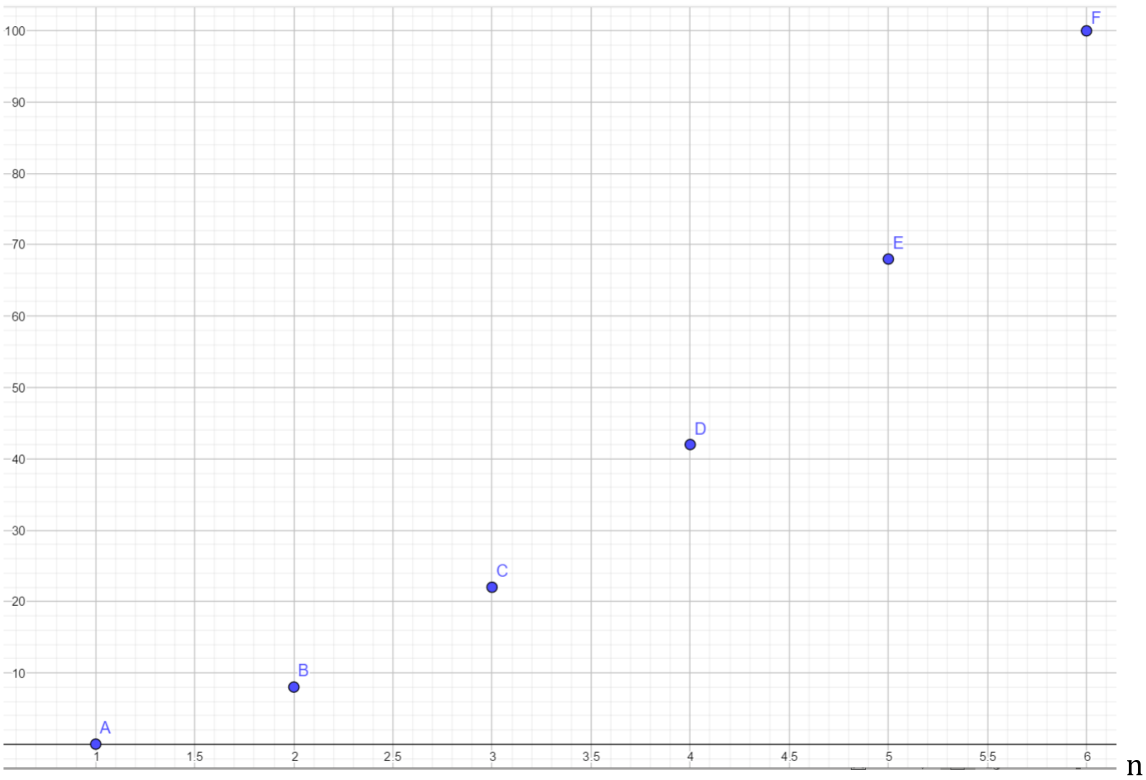
8 14 20 26 32

6 6 6 6

(b) As can be seen from the pattern above, the second difference is a constant equal 6.

(c)

T_n The graph of T_n vs n.



A close look at the position of each point on the graph indicates the quadratic nature of the graph passing through these points. This can be good enough evidence that the sequence $a_n = 3n^2 - n - 2$ is **quadratic** in nature.

Diagnostic Assessment This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

The table below represents the sequence

5, 11, 21, 35

x	y
1	5
2	11
3	21
4	35

What is the formula for this?

a. $2x^2$

b. $2x^2 + 5$

c. $2x + 4$

d. $2x^2 + 3$

The correct answer and possible misconceptions:

A. Wrong answer. Probably guessing.
B. Wrong answer. Probably guessing.
C. Wrong answer as this is linear
D. This is the CORRECT answer:

https://diagnosticquestions.com

Why do this activity?

Sequencing is a strategy that promotes reasoning; reasoning on the other hand is a very important strand within Mathematics as a subject, and it is important as a wider **thinking skill** that can be developed through learning Mathematics.

The topic helps children develop skills of explaining/ justifying why an answer to a given problem must be correct. As children grow deeper in understanding sequences, they develop skills of constructing chains of deductions based on the nature of the sequences.

Familiarity with non-linear sequential data can support work on modelling. There is also development of automatization skills in dealing with patterned and sequential data.

Children's 'algebraic skills' are further sharpened in this kind of activity. Quadratic sequences are known for their power to predict some future events.

Learning objectives

In doing this activity children will have an opportunity to:

- **Investigate** whether a given numeric sequence is **quadratic** in nature.
- **Generate** a sequence given the general formula, T_n.
- **Determine** the n^{th} term of a numeric sequence.
- **Plot a graph** of a given sequence, in terms of the terms T_n of the sequence versus n, the position of each term in the sequence.

Generic competences

In doing this activity students will have an opportunity to:

- **think mathematically,** reason logically and give explanations and develop effective communication skills;
- **think flexibly**, be creative and innovative and apply knowledge and skills;
- **visualize** and develop the skill of interpreting and creating visual images to represent concepts and situations;
- interpret and **solve problems** in a variety of situations;
- **work and learn independently** and prepare for lifelong learning;
- **work in a team**:
  - collaborate and work with a partner or group
  - have empathy with others, listen to different points of view
  - develop leadership qualities;
- **communicate** in writing, speaking and listening according to the audience:
  - exchange ideas, criticise, and present information and ideas to others
  - analyze, reason and record ideas effectively;
- **develop life skills and consideration for others -** to show social responsibility – to work for the good of the community.

Suggestions for teaching

If you have one or more children, organise them into small groups to promote group discussions, although the work can also be done on an individual basis. Ask prompts geared at determining their algebraic skills on substituting in each formula given, evaluating numeric values in an expression. Take children through the Diagnostic Assessment prior to engaging them with the activities on the HELP and NEXT sections of this whole lesson activity.

Key questions

1. Can you compute terms of a quadratic sequence given the general formula or the n^{th} term?
2. Can you explain how you would justify that a given numeric sequence is quadratic?
3. Can you demonstrate graphically that a given sequence is quadratic in nature?
4. Can you come up with the formula which generates a given quadratic sequence?
5. How do you prove that this formula is correct?

Follow up

1. https://aiminghigh.aimssec.ac.za/tag/sequences/
2. https://aiminghigh.aimssec.ac.za/wp-content/uploads/2018/01/LS-A2-Sequences-and-Patterns.pdf

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. New material will be added for Secondary 6.
For resources for teaching A level mathematics see https://nrich.maths.org/12339
Note: The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is beyond the school curriculum for Grade 12 SA.

	Lower Primary or Foundation Phase Age 5 to 9	Upper Primary Age 9 to 11	Lower Secondary Age 11 to 14	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6