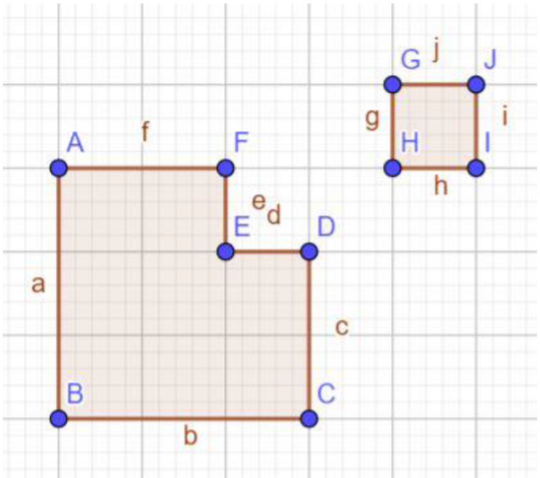
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

GRAPHING QUADRATICS

Graph (sketch) $y = x^2 - 2x - 1$.
What type of an equation is this?
Remember the general form of an equation of this nature: $y = ax^2 + bx + c$
When $a > 0$ the graph is ∪ – shaped and when $a < 0$ it is ∩ – shaped.
What is the situation in our problem?
Does the graph of $y = x^2 - 2x - 1$ intersect with the x-axis?
What is the value of y when this graph crosses the x-axis?
Can you solve $x^2 - 2x - 1 = 0$?



Think of the method of completing a square for solving quadratics.

Geometrically, you are thinking of adding shape GHIJ to the bigger shape to make it a complete square, then solving the problem. What two values of x are solutions to $x^2 - 2x - 1 = 0$?

From $x^2 - 2x - 1 = (x - 1)^2 - 2 = 0$, notice that the graph is above the x-axis for $(x - 1)^2 \geq 0$ for all values of x. The minimum value of $(x - 1)^2$ is zero and this happens $x - 1 = 0$, that is $x = 1$. What is the value of y when $x = 1$?
Comment on this point.

Parabolas are symmetrical shapes, what is the line of symmetry of this graph?

At what point(s) does the graph cross the x-axis?

Taking into consideration all the special points you have obtained so far, draw and label axes and plot all the special points.

Using the special points sketch the graph of $y = x^2 - 2x - 1$.

HELP

It's important to find the coordinates of special points, that is points where the graph cuts the y–axis (when $x = 0$ which is easy to find) and where it cuts the x–axis (when $y = 0$ corresponding to the solution of an equation).

The other special points are where the graph 'changes direction', where it has a maximum or minimum value (which you can often spot from the symmetry of the graph) and also where the graph has an S-bend (called a point of inflection).

NEXT

It is necessary for you to learn to sketch graphs from your knowledge of the type of equation without first making a table of values and plotting points from the table. This is a skill you will need as you go on with your studies in mathematics and science, for example in topics like calculus and trigonometry.

(1) Complete the square in this quadratic expression: $f(x) = x^2 - 4x - 1$ and find the solution of $x^2 - 4x - 1 = 0$.
(2) Find the coordinates of the turning point of $f(x)$ and the point where the graph cuts the y-axis.
(3) Sketch the graph of the function without using a table of values or graphing software.
(4) If you have the software on your device you can use GeoGebra or some other graphing software to check your work.

Read the following if you need help.
To determine whether the graph is ∪ shaped with a minimum turning point or ∩ shaped with a maximum turning point try the following:

In our problem, the quadratic function is $y = f(x) = x^2 - 2x – 1 = (x - 1)^2 – 2$.
We saw that it is a minimum when $x = 1$ and $f(1) = 1^2 - 2(1) – 1 = -2$.
So (1 ; -2) is a **minimum point and the graph is ∪ shaped.**

The function $f(x) = -(x)^2 + 2x - 1 = -(x – 1)^2 \geq 0$ for all x and $f(x) = 0$ for $x = 1$.
So (1 ; 0) is a **maximum point and the graph is ∩ shaped.**

In general, the completed square form of a quadratic expression $f(x) = a(x + p)^2 + q$ can help sketching quadratic graphs if we understand that the turning point (-p, q) occurs when $x = -p$, and if we understand that we can decide whether it is a maximum or a minimum depending on the sign of the coefficient a.

If $x + p = 0$ and $a > 0$ then then (-p, q) is a minimum and the graph is ∪-shaped.
If $x + p = 0$ and $a < 0$ then then (-p, q) is a minimum and the graph is ∩-shaped.

NOTES FOR TEACHERS

SOLUTION

(1) The quadratic function y = $f(x)$ = x^2 - 2x – 1 = $(x - 1)^2 - 2$ is a parabola.

The graph intersects the x-axis when y = 0.

Remember from the quadratic formula: $x = [-b \pm \sqrt{(b^2-4ac)}]/\ 2a$

If the discriminant $b^2 - 4ac \geq 0$, there are two distinct places where the graph intersects with the x-axis.

Solving $x^2 - 2x - 1 = 0$ we need to think of completion of a square method.

What can we add to the expression on the left hand side to make it a perfect square?

$$x^2 - 2x - 1 = 0$$

Adding 1 to both sides: $x^2 - 2x = 1$.

Completing the square by adding 1 again: $x^2 - 2x + 1 = 2$

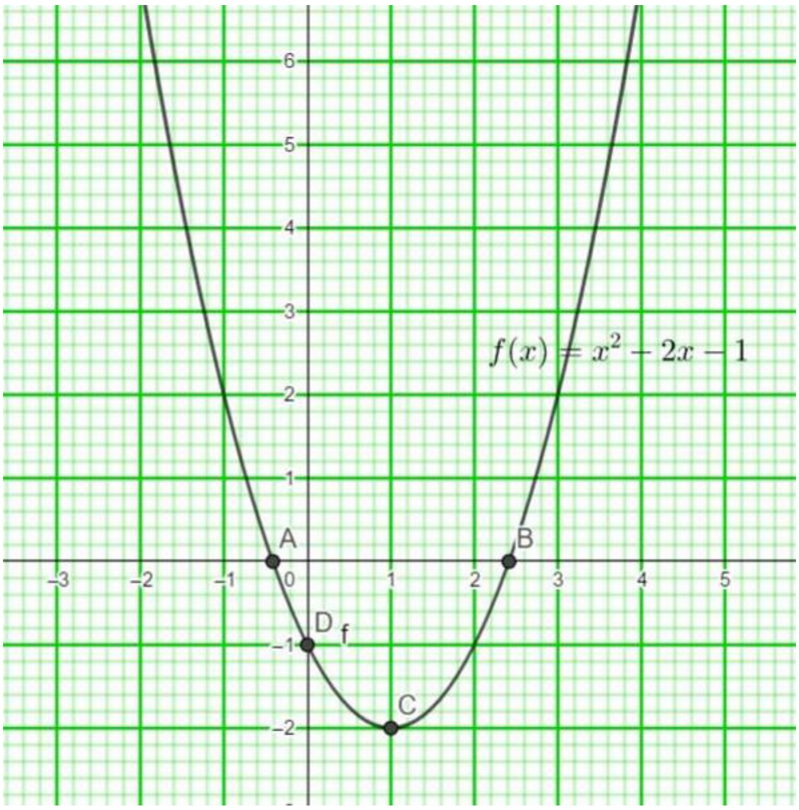
So $(x - 1)^2 = 2$

Taking the square root of both sides: $(x - 1) = \pm\sqrt{2}$

$$x = 1 \pm\sqrt{2}$$

The points where the graph cuts the x=axis, called the zeros or roots of the equation, are (to 3 decimal places):

$$x = -0.414 \text{ and } x = +2.414.$$



(2) The minimum value of $(x-1)^2$ is zero, this happens when $x = 1$, and $f(1) = 1^2 - 2(1) - 1 = -2$, so the point (1 ; -2) is a minimum point and the graph is ∪ shaped.

Notice that the point (1 ; -2) is half way between the zeros $x = 1 \pm\sqrt{2}$ of the function and the vertical line $x = 1$ is a line of symmetry for the graph.

The graph is above the x -axis for all values of $x \leq -0.41$ and ≥ 2.41.

(3) The special points (written as decimal approximation so that they can be plotted) are: A = (-0.41; 0) and B = (2.41 ; 0) where the graph cuts the x – axis

C = (1 ; -2) the turning point (a minimum)

D = (0 ; -1) where the graph cuts the y-axis.

(4) The general form of quadratic function is y = $f(x)$ = ax^2 + bx + c.

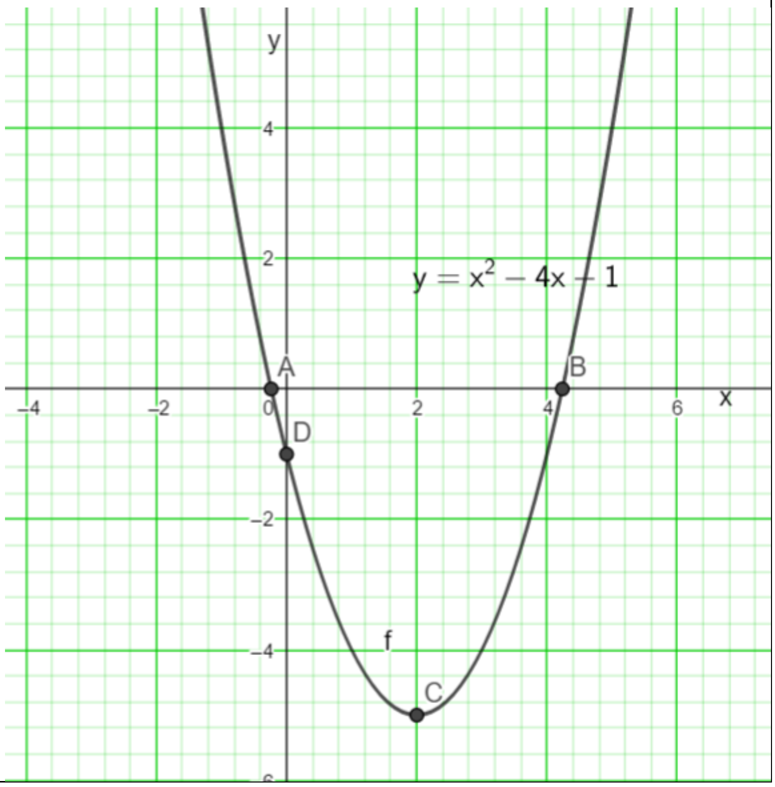
Completing the square gives $f(x) = a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + c$

So $f(x) \geq -\left(\frac{b}{2a}\right)^2 + c$ for all values of x.

The point $x = \frac{-b}{2a}$, $y = -\left(\frac{b}{2a}\right)^2 + c$ is a minimum point, the graph is ∪ shaped and the line x =$\frac{-b}{2a}$ is a line of symmetry.

NEXT

The graph of the function
$f(x) = x^2 - 4x - 1$
cuts the x =axis when
$x^2 - 4x - 1 = 0$.
To solve this equation first add 1 to both sides:
$x^2 - 4x = 1$
So $(x - 2)^2 = 1 + 4 = 5$
Giving the solutions:
$x = 2 \pm \sqrt{5}$



Why do this activity?

This activity links the process of solving quadratic equations with a careful analysis of the properties of the graphs of quadratic functions (parabolas). It explores the special points: the turning points (maxima, minima and points of inflection), the roots/zeros of the graph (x -intercepts), the y-intercept(s) and the line of symmetry which are all essential to the understanding of simple functions.

This activity explores the following methods of solving quadratic equations, and it explores the connections between them:

- factorization;
- completing the square;
- the use of the quadratic formula (which is derived by completing the square);
- graphical methods.

When they first meet graphs, young learners make a table of values, then plot the points from the table on a Cartesian plane. By about halfway through secondary school they should have grown out of this phase and have developed the skill of recognising types of graphs from their equations, also finding points where the graphs cut the axes and finding the turning points (not by calculus but by inspection of the equation). Calculus is a faster method of finding the special points but, to understand functions learners need to know and understand all the methods and the connections between them.

This activity also gives a proof of the quadratic formula and explains how it arises from the completion of a square method.

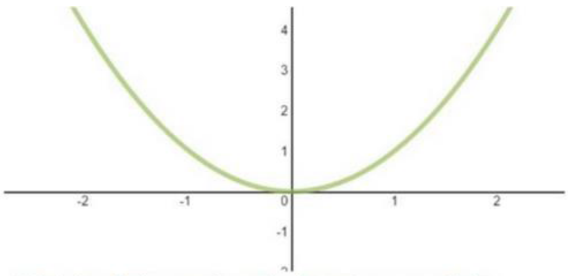
Learning objectives

In doing this activity students will have an opportunity to:

- review work on the use of the quadratic formula to solve equations;
- review work on the completion of the square method to solve quadratic equations;
- find the mean of two numbers (the of two x-intercepts; to find the line of symmetry of the graph.
- evaluate a function for given values of x, e.g. find $f(2)$ for a given function y = $f(x)$

DIAGNOSTIC ASSESSMENT This should take about 5–10 minutes.

Write the question on the board, say to the class:
"Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 for D".



Which of the options below is a possible equation for the above graph?

A $y = x^3$ B $y = x^2$

C $y = 1/x$ D $y = x$

1. Notice how the learners respond. Ask a learner who gave answer A to explain why he or she gave that answer. DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
2. It is important for learners to explain the reasons for their answers. Putting thoughts into words may help them to gain better understanding and improve their communication skills.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. Ask the class to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

The correct answer is: **B**
Possible misconceptions:
Any other answer given could be because the learners are guessing, especially with regards to A and C if they are not yet familiar with a cubic graph and a hyperbola respectively. Learners who chose C did not understand linear graphs. https://diagnosticquestions.com

Generic competences

In doing this activity students will have an opportunity to:

- **think mathematically,** reason logically and give explanations and proofs;
- **think flexibly**, be creative and innovative and apply knowledge and skills;
- develop the **skill of visualization** and interpreting and creating visual images to represent concepts and situations;
- **communicate** in writing, speaking and listening and:
  - exchange ideas, criticise, and present information and ideas to others;
  - analyze, reason and record ideas effectively.

Suggestions for teaching

Resources: Graph paper, 30-centimetre ruler, pencil, pencil eraser, scientific calculator.

The teacher may start by engaging learners in a discussion of what they know about quadratic equations and how they are related to quadratic functions and their graphs. By asking questions and encouraging learners to explain their ideas, the teacher carries out diagnostic assessment while, at the same time prompting the learners to remember what they have learnt in the past.

Give out copies of the worksheet on page 1 or write the question on the board. Encourage the learners to work in pairs and to talk help each other. If learners are struggling to get started give them the HELP worksheet.

After 20 minutes or so organise the class so that learners work in groups of four. From time to time hold a class review of 'where we are up to' when you invite some individuals (or pairs) to come to the front of the class and explain what they have done.

This work will almost certainly take 2 lessons and a homework between. Learners who finish before the teacher wants to wind up the lesson can be given the NEXT worksheet.

During these lessons, in several whole class discussions, and by engaging learners in 'reporting on their work to the class', the teacher should guide the learners through the solutions as outlined in the SOLUTIONS Section.

At the end of the second lesson the teacher might give a summary and the class can make notes in a summary of what they have learned.

Key questions

1. Did you meet any new words today? If so, which word(s)? What is the meaning of the word(s)?
2. Can you explain, in your own words, all the necessary steps leading to sketching a parabola of shape ∪?
3. Can you follow the same steps for sketching a quadratic equation of the shape ∩ ?

Follow up

Quadratic Matching 1 & 2 https://aiminghigh.aimssec.ac.za/quadratic-matching-1/
https://aiminghigh.aimssec.ac.za/quadratic-matching-2/
Quadratic Equations https://aiminghigh.aimssec.ac.za/quadratic-equations/
Quadratic Functions https://aiminghigh.aimssec.ac.za/quadratic-functions/