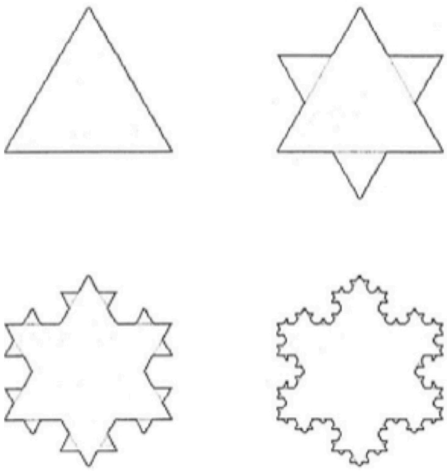
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE
TEACHER NETWORK

THE KOCH CURVE

Image starting with an equilateral triangle and replacing each edge by a zig-zag curve made up of 4 pieces. Each of the 4 pieces is one third of the length of the line segment it replaces so it looks as if equilateral triangles have been attached to the shape. Now imagine repeating this process over and over again. What would happen?

The edges of the triangle at the first stage are one unit in length. Here are the first four stages.



The Koch curve is sometimes called the snowflake curve. This curve is the outer perimeter of the shape formed by the outer edges when the process is repeated infinitely often.

1. The table shows that the snowflake construction produces three types of sequences A, B and C. The last row gives the *n*th term in the sequence. Describe the type of sequence in each case giving two properties of the sequence.

2. What happens to the perimeter as n tends to infinity?

Koch snowflake	A Length of edge	B Number of edges	C Perimeter
1	1	3	3
2	$\frac{1}{3}$	12	4
3			
4			
5			
n			

The next table shows some of the calculations of the area enclosed by the Koch curve.
3. Show that the **sum of the increases in area** in column C is

$$\frac{1}{3}\left(1+\frac{4}{9}+\left(\frac{4}{9}\right)^{2}+\cdots+\left(\frac{4}{9}\right)^{n-2}\right)$$

Find the sum of this series.

In studying what happens to the area inside the curve in this iteration it is convenient to take the area of the original triangle as 1 square unit for the calculations. To relate this calculation to the previous table where the edge of the triangle has length 1 unit use the scale factor √3/4.]

Investigate the increase in area of the Von Koch snowflake at successive stages. Call the area of the original triangle one unit and complete the table below.

Koch snowflake	A Area of each additional triangle	B Number of additional triangles	C Increase in area
1	1		
2	$\frac{1}{9}$	3	$3\left(\frac{1}{9}\right)$
3	$\left(\frac{1}{9}\right)^2$	12	$3(4)\left(\frac{1}{9}\right)^2=\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)$
4	$\left(\frac{1}{9}\right)^3$	48	$3(4)^2\left(\frac{1}{9}\right)^3=\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)^2$
5	$\left(\frac{1}{9}\right)^4$	192=3(4)3	$3(4)^3\left(\frac{1}{9}\right)^4=\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)^3$
n	$\left(\frac{1}{9}\right)^{n-1}$		

4. What happens to the sum of the increases in area as n tends to infinity?

5. The difference between what happens to the perimeter and to the area of the Koch snowflake curve as n tends to infinity is very interesting. Comment on this difference.

SOLUTION

Fractals are infinite. To understand fractals we look at a sequence of stages in the repetitive process that leads to the fractal in the limit. Here we want to find the length of the von Koch curve fractal and the area it encloses.

1. The table shows that, in calculating the length of the Koch snowflake, we get three types of sequences A, B and C.

A Geometric sequence, first term 1, common ratio 1/3
B Geometric sequence, first term 3, common ratio 4
C Geometric sequence, first term 3, common ratio 4/3

2. The length of the Koch snowflake curve (the perimeter of the shape) at the nth stage is: $3\left(\frac{4}{3}\right)^{n-1}$.

The perimeter tends to infinity because 4/3>1 .

So the length of the fractal curve is infinite.

Von Koch snowflake	A Length of edge	B Number of edges	C Perimeter
1	1	3	3
2	$\frac{1}{3}$	12	4
3	$\left(\frac{1}{3}\right)^2$	$48=3(4)^2$	$3\left(\frac{4}{3}\right)^2$
4	$\left(\frac{1}{3}\right)^3$	$192=3(4)^3$	$3\left(\frac{4}{3}\right)^3$
5	$\left(\frac{1}{3}\right)^4$	$768=3(4)^4$	$3\left(\frac{4}{3}\right)^4$
n	$\left(\frac{1}{3}\right)^{n-1}$	$3(4)^{n-1}$	$3\left(\frac{4}{3}\right)^{n-1}$

Total **increase** in area = $T=\frac{1}{3}+\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)+\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)^2+\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)^3+\dots+\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)^{n-2}$

$$=\frac{1}{3}\left[1+\left(\frac{4}{9}\right)+\left(\frac{4}{9}\right)^2+\left(\frac{4}{9}\right)^3+\dots+\left(\frac{4}{9}\right)^{n-2}\right]$$

4. What happens to the sum of the increases in area as n tends to infinity?

As n $\to\infty$ the total increase in area $\to\left(\frac{1}{3}\right)\left(\frac{1}{1-\frac{4}{9}}\right)=\frac{3}{5}$

So the area enclosed by the fractal curve is 1 unit plus the area added on so the area enclosed tends to 1.6 units.

5. The difference between what happens to the perimeter and to the area of the Von Koch snowflake as n tends to infinity is very interesting.

As n tends to infinity the perimeter tends to infinity but the area enclosed remains finite and it tends to 1.6 units.

Generalising this idea to 3 dimensions to give a FINITE VOLUME enclosed by a surface that has VERY LARGE AREA explains why so many structures in nature are fractal in form although they are not truly fractal because they are not infinite. As examples we have trees, root systems, and our lungs and vascular system (veins and arteries). The large surface area is optimal for the exchange of gases or (in the case of root systems) of nutrients, across the surface.

NOTES FOR TEACHERS

Diagnostic Assessment

This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

The first term of a geometric sequence is 6 and the common ratio is 3.
Find the sum of the first 5 terms of the sequence.

A. 240 B. 1452
C. 726 D. 484

The correct answer a is C: 6 + 18 +54 + 162 + 486 + 726
or 6(243 -1)/(3-1) = 242 x 3 = 726
Possible misconceptions:
Used formula incorrectly
A. 6 x (1 - 3 to the power of 4 / 1 - 3)
B. 6((3^5)-1)
D. 6*3^(5-1)
https://diagnosticquestions.com

Why do this activity?

This is just an activity about summing finite and infinite geometric series but it links to other topics in mathematics and in science, nature and art. The iterative process that builds the fractal involve terms in several different geometric series, and students can see the process unfolding and imagine metrically and algebraically.

Learning objectives

Summing finite and infinite geometric series.
Experience of the applying of mathematics to other fields of study and to the natural world.

Generic competences

Visualisation.
Ability to apply ideas from one discipline to another.

Suggestions for teaching

Start by drawing an equilateral triangle on the chalkboard. Then explain that the Koch curve will be produced by a repeating process that can be visualised in 2 ways:

(1) By replacing all the line segments at each stage by a zig-zag line made up of 4 pieces of equal length
(2) By attaching an equilateral triangle to each line segment that has edge length one third of the length at the previous stage, and focussing on the outer edge of the shape formed.

Students might work in pairs. Take each of the 5 questions one at a time, and have the learners coming to the board to explain what they have done on each part before moving on to the next part, and have a class discussion on each one

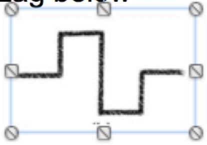
Key questions

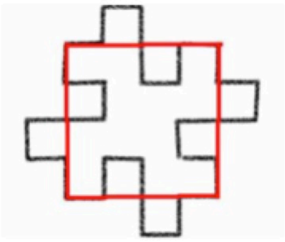
(1) What happens to the length of the line segments at each stage?
(2) What is the scale factor between the lengths from one stage to the next?
(3) How can you work out how many triangular protrusions are added to the shape at each stage?
(4) What do you notice about the terms of that sequence?
(5) What happens to that value as n tends to infinity?

Possible extension

Investigate the length of the squareflake fractal curve, and the area it encloses, by considering the limit as n tends to infinity of the length at the nth stage.

1. Start with a square (stage 0)
2. Replace each edge with the zig-zag below



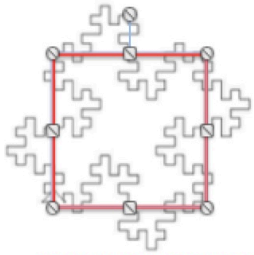


Squareflakes stage 0 (red) and stage 1

3. Repeat step 2 infinitely often.

What happens to the area?

How does the perimeter change?



Squareflake stage 2

Possible support

You might use the 1 – 2 – 4 - more teaching strategy for group work. Pair students so that a more able student works with each student who might find difficulties, but tell them which one of them will be asked to speak for the pair, and to explain their work to the class. When most pairs have finished each section, have the class work in fours to compare and check answers.

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. New material will be added for Secondary 6.
For resources for teaching A level mathematics see https://nrich.maths.org/12339
Note: The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is **beyond** the school curriculum for Grade 12 SA.

	Lower Primary or Foundation Phase Age 5 to 9	**Upper Primary Age 9 to 11**	**Lower Secondary Age 11 to 14**	**Upper Secondary Age 15+**
South Africa	**Grades R and 1 to 3**	**Grades 4 to 6**	**Grades 7 to 9**	**Grades 10 to 12**
USA	**Kindergarten and G1 to 3**	**Grades 4 to 6**	**Grades 7 to 9**	**Grades 10 to 12**
UK	**Reception and Years 1 to 3**	**Years 4 to 6**	**Years 7 to 9**	**Years 10 to 13**
East Africa	**Nursery and Primary 1 to 3**	**Primary 4 to 6**	**Secondary 1 to 3**	**Secondary 4 to 6**