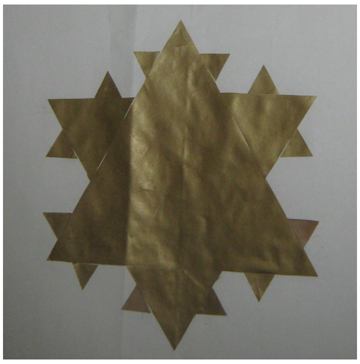
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE
TEACHER NETWORK

MAKE A VON KOCH CURVE

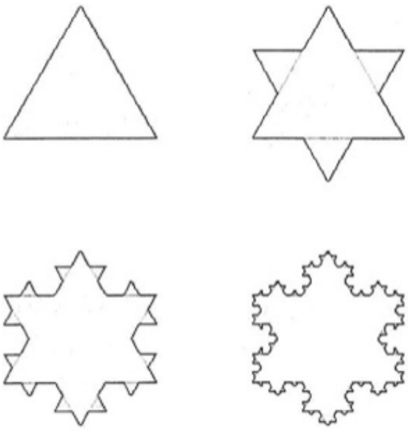


The curve, also known as the 'snowflake curve' was invented in 1904 by the Swedish mathematician Helge von Koch.

Take a large piece of backing paper, and either draw an equilateral triangle with edges of length 27 cm or cut a triangle from coloured paper and stick it on a backing sheet.

Step 1 Divide the triangle edge length by 3.
Step 2 Make a triangle with this edge length for each edge of the previous shape.
How many triangles do you need? Record this number each time you go through these instructions.
Step 3 Stick one new triangle to the middle of each of the edges of the previous shape.

Repeat Steps 1 to 3 until you have made triangles that are 1cm edge length, and stuck them on.
Alternatively you can complete the last stage by drawing 48 triangles with edge length 1 cm onto the shape.



Imagine these steps being repeated again and again many times even though the triangles would actually get too small to handle.

The diagram shows Stages 0, 1, 2 and 3 of the von Koch curve, not fractals but steps in the creation of the fractal that would result if the process were to be repeated infinitely often. The outer edge of the snowflake pattern is called the von Koch curve.

The poster above is at Stage 2.

Record the lengths of edges you have made.

Can you predict what the lengths of edges of the next few sets of triangles would be?
[It is better to use fractions rather than decimals].

Draw and fill in a table like this to find the perimeter at each stage of the von Koch curve. Add as many stages as you can.

Stage	Number of triangles added during this step	Length of edge of triangles added	Total number of edges	Total perimeter of von Koch curve in cm
0	1	27	3	
1	3		12	
2				

SOLUTION

Stage	Number of triangles added during this step	Length of edge of triangles added	Total number of edges	Total perimeter of von Koch curve in cm
0	1	27	3	81
1	3	9	12	108
2	12	3	48	144
3	48	1	192	192
4	192	1/3	768	256
5	768	1/9	3072	341.3
…				
n	$3\times4^{n-1}$	$27(1/3)^n=(1/3)^{n-3}$	3×4^n	$81\times(4/3)^n$

The perimeter gets longer and longer at each stage. For example, at the 50th stage the perimeter is 1430 kilometres. At the 70th stage the perimeter is 451 020 kilometres which is greater than the distance from the earth to the moon. At the 91st stage 190 million kilometres which is greater than the distance between the Earth and the sun.

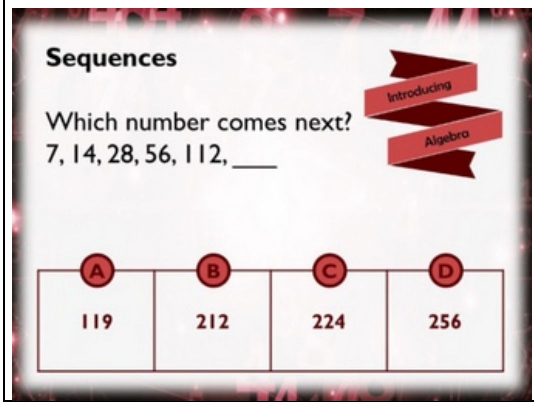
In contrast, if you draw a circle of radius 13 cm around the von Koch shape at any stage the whole shape will fit inside the circle. Alternatively the von Koch shape at any stage fits inside a square of edge length 24 cm. The area of the von Koch fractal is 1.6 times its area at the start, that is 505 cm^2 to the nearest cm^2 for the von Koch shape you have made.

NOTES FOR TEACHERS

Diagnostic Assessment

This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.



The correct answer is C because 2 × 112 = 224
and possible misconceptions:

A. Students have added 7 perhaps because looking only at the first two terms of the sequence.

B. Students have added 100 or guessed.

D. Probably these students just guessed.

https://diagnosticquestions.com

Why do this activity?

This activity provides a different approach to number patterns and introduces simple geometric sequences (multiplying by 4 or dividing by 3). The activity also links algebra to geometry and provides an exercise in working out the perimeter of a shape. It also provides an introduction to the idea of a limit in mathematics with students actually starting, in a practical concrete way, a process that, theoretically, goes on forever. By some simple calculations, without infinite series, they can see that the perimeter grows very quickly while the area grows hardly at all. There are real world applications in the fractal structure of the lungs and vascular system, trees and roots etc.

Learning objectives

Practice in working with number patterns leading to geometric sequences.
Review of, (or for young children, introduction to) the concept of perimeter
Review of, (or for young children, introduction to) the concept of area
For older students: practice in using a calculator and calculations involving powers.
Appreciation of the connections between this topic and nature and human biology.

Generic competences

Creativity, teamwork, ability to work systematically, appreciation of applications of mathematics and the wonders of nature.

Suggestions for teaching

Resources: Backing sheet, glue, equilateral triangles cut out of coloured paper: one with edge length 27 cm, 3 with edge length 9 cm, 12 with edge length 3 cm and 48 with edge length 1 cm. Alternatively you can complete the last stage by drawing 48 triangles with edge length 1 cm onto the shape. Calculators for secondary students.

Start with the diagnostic question and a class discussion about number patterns that arise from doubling. For learners, this connects the known to the unknown that will follow in the lesson.

Have the backing sheet pinned up before the lesson and get all the learners involved in making the poster by giving each learner a triangle to stick on.

At each stage stop, ask the learners what they notice. **For very young learners this will be sufficient.**

For older primary students and secondary students you could have the table for the results on the board and, before going on to make the next stage for your poster, the class could fill in each row for the numbers of triangles added, the length of the edges of the triangles, the total number of edges and the perimeter.

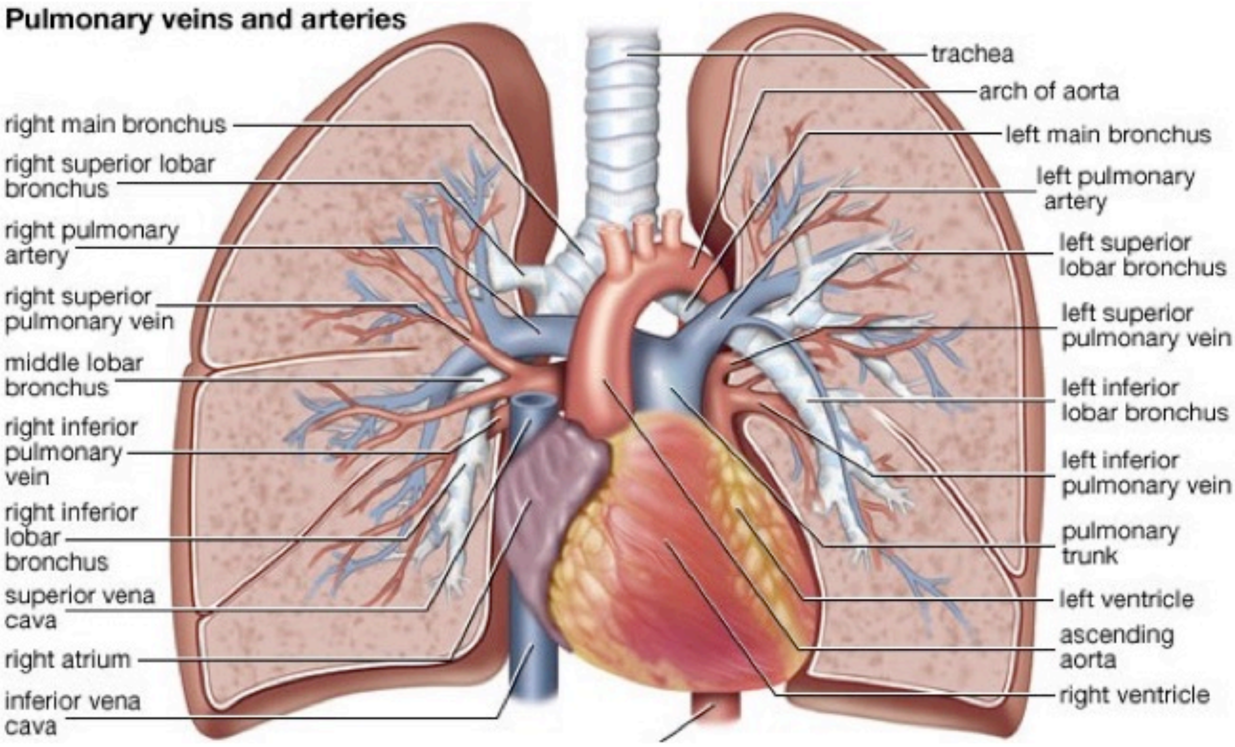
When the poster is complete and the table is filled in, ask the class what they notice about the first column of numbers and what they think the next numbers will be in that column. It is very likely that some learners will say that the pattern starts to go up in multiples of 4. If no learners suggest this, then ask them if they see any connection with the example 7, 14, 28, 56, 112 … that started the lesson.

For each column in turn ask what the students notice and discuss the number patterns and how the sequences will continue for the next few terms. Ask what they think will happen if the steps are repeated infinitely often.

For secondary students: ask the students to use calculators to work out the next few rows in the table, and ask them for the rule for continuing each sequence that would give them the entry in the table for say the 10^{th} or the 20^{th} stage without completing all the rows in between. Older or more able students can be asked to give the rules the nth terms.

For all students

(1) Talk about the perimeter growing so that soon it is longer than the distance from the Earth to the moon, then it keeps increasing until it is longer than the distance from the Earth to the sun and keeps increasing to infinity.
(2) Talk about the area growing very very slowly and staying inside a square of edge length 24 cm. Draw this square on your poster.
(3) Tell them that our 3D lungs are like the 2D von Koch shape, so our lungs have a huge surface area but a volume that fits inside our chests. Talk about how we breathe in oxygen to our lungs, the oxygen passes across the surface of our lungs into our bloodstream through the fractal system of blood vessels, then is pumped around the body.



(4) Get the students to breathe in then breathe out and think about what happens to the air that goes in and out of our lungs. Ask what happens as they breathe out. Show the diagram and explain why, as well as the surface of our lungs being fractal, our blood supply system (arteries and veins) are also fractal in form.

Key questions

(1) What happens to the number of triangles added at each stage?
(2) What do you notice about that number pattern?
(3) Can you give a rule for working out how many triangles are added at each stage.
(4) What happens to the size of the triangles added at each stage?
(5) Do the triangles added get smaller or bigger?
(6) What happens to the lengths of the line segments around the edge of the shape at each stage?
(7) Can you give a rule for working out the edge length of the triangles are added at each stage?
(8) How can we work out the perimeter at each stage?
(9) What is the scale factor between the lengths of the line segments from one stage to the next?
(10) How long do you think the perimeter will be at the 50th stage? Could we work it out.
(11) The distance from the earth to the moon is 384 400 km. Do you think the perimeter at the 50th stage is less than this or more than this distance to the moon?
(12) The distance from the earth to the sun is 150 million km. Do you think the perimeter at the 100th stage is less than this or more than this distance to the sun?
(13) If we draw a square of edge length 23 cm do you think that the whole von Koch shape will stay inside that square as it grows into an infinite fractal?

Possible extension

You can ask similar questions about area and fill in a similar table for area to the table for distance.
For older students who are learning about summing geometric series see
https://aiminghigh.aimssec.ac.za/years-10-to-12-the-von-koch-curve/

Possible support

As this activity is visual and all students, of any age, can be given triangles to stick on the poster, younger children, and those who struggle with maths, will be able to see the fractal shape changing from stage to stage. The number work can be introduced at levels suitable for the individual students. The aim should be for all students to get the general idea of a shape growing to a pattern according to given rules, that the perimeter gets infinitely long and the area does not increase very much. You also want all students to have some appreciation of the connection with our lungs and blood vessels.

<table>
<tr><td colspan="5">Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. New material will be added for Secondary 6.
For resources for teaching A level mathematics see https://nrich.maths.org/12339
Note: The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is beyond the school curriculum for Grade 12 SA.</td></tr>
<tr><td></td><td>Lower Primary
or Foundation Phase
Age 5 to 9</td><td>Upper Primary

Age 9 to 11</td><td>Lower Secondary

Age 11 to 14</td><td>Upper Secondary

Age 15+</td></tr>
<tr><td>South Africa</td><td>Grades R and 1 to 3</td><td>Grades 4 to 6</td><td>Grades 7 to 9</td><td>Grades 10 to 12</td></tr>
<tr><td>USA</td><td>Kindergarten and G1 to 3</td><td>Grades 4 to 6</td><td>Grades 7 to 9</td><td>Grades 10 to 12</td></tr>
<tr><td>UK</td><td>Reception and Years 1 to 3</td><td>Years 4 to 6</td><td>Years 7 to 9</td><td>Years 10 to 13</td></tr>
<tr><td>East Africa</td><td>Nursery and Primary 1 to 3</td><td>Primary 4 to 6</td><td>Secondary 1 to 3</td><td>Secondary 4 to 6</td></tr>
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