The Game of NIM

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1 Introduction

The game of NIM is a game for **two players** and the rules are as follows:
(1) The game starts with a number of piles of stones;
(2) the first player removes some (at least one, and possibly all) of the stones from **one** (and only one) pile;
(3) the second player does the same;
(4) the first player plays again, and so on.
The player who takes the last stone is the winner.

We can describe this game by giving the starting position as a list of of positive integers. For example, we can represent the the starting position which has three piles of stones, with 41, 163 and 13 stones in the three piles, by the vector $(41, 163, 13)$. Suppose that the two players are A and B, and that A starts the game by removing 16 stones from the first pile: we can represent this move by

$$(41, 163, 13) \xrightarrow{A} (25, 163, 13)$$

Thus the game might be played as follows, with players A and B, with A starting (and B winning):

$$(41, 163, 13) \xrightarrow{A} (25, 163, 13) \xrightarrow{B} (25, 19, 13) \xrightarrow{A} (25, 13, 13) \xrightarrow{B} (0, 13, 13)$$

$$\xrightarrow{A} (0, 7, 13) \xrightarrow{B} (0, 7, 7) \xrightarrow{A} (0, 0, 7) \xrightarrow{B} (0, 0, 0)$$

The **analysis of the game** depends on writing the numbers of stones in binary form [1] which we do as follows:

$$\begin{pmatrix} 41 \\ 163 \\ 13 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix};$$

[1] In Octave: `dec2bin(41) = 101001, dec2bin(163) = 10100011, dec2bin(13) = 1101`

for example, $41 = 1 + 8 + 32 = 2^0 + 2^3 + 2^5$. The next step is to form the sum of each column:

$$\begin{pmatrix} 41 \\ 163 \\ 13 \\ -- \\ \text{sum} \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - \\ 3 & 1 & 1 & 2 & 0 & 2 & 0 & 1 \end{pmatrix};$$

We say that the game is an **even position** if each of the column sums is an **even integer**; otherwise it is an **odd position**. For example, for the position $(41, 163, 13)$ given above is an odd position.

Obviously, in any even position there are **at least two non–empty piles**, and this means that if you (as a player) receives an even position and then has to remove some of the stones, then you **cannot** win at this turn (because you can remove stones from only one pile). Thus if you are playing, your aim must be pass on a even position to your opponent whenever possible. In fact, if you **always** pass on an even position then your opponent can never win, **so you will win**.

The strategy for playing depends on the following two facts:

1. if you pass an even state to your opponent then
   they can only pass an odd state back to you;

2. if you receive an odd state from your opponent then
   you can always pass an even state back to your opponent.

We now argue as follows. Suppose that the game is in an odd state and that it is your turn to play. By (2) you can remove some stones and pass an even state to your opponent. Your opponent plays and by (1), they will pass an odd state back to you. Then you play and pass an even state back to your opponent; they pass an odd state back to you, and so on. Now it is clear that if a player receives an even state, then they cannot win at the next turn. Thus **with this strategy, your opponents will never win. As the game must finish, you will win**.

The proofs of (1) and (2)

Suppose that we receive an even position. We remove some stones and so there is some coefficient, say a_k that changes (from 1 to 0, or from 0 to 1). As the sum of the coefficients of 2^k taken over all piles was originally even (we received an even position), for some column it will now be odd, so the resulting position will be odd. This proves (1).

To prove (2) express each number (in each pile) in binary form and find the largest k for which the sum of the k–th coefficients is odd. We shall remove stones from this pile, and it is easiest to decide how many stones to remove if we think of removing all stones from this pile and then replacing some. In fact, we remove all stones from the pile (that is, we temporarily convert all of the binary coefficient for this pile to 0) and then, for each n, we replace stones (i.e. convert the binary coefficients back to 1) in order to make each sum of the coefficients an even integer. This proves (2).

An example to explain (2) may help. Suppose that the position is has three piles in the form

$$(25, 28, 10) = ([1, 0, 0, 1, 1], [0, 0, 1, 1, 1], [0, 1, 0, 1, 0]).$$

This is an odd position, and it is perhaps easiest to see if we write it in the form of a matrix:

$$\begin{pmatrix} 25 \\ 28 \\ 10 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

The largest k for which the sum of coefficients of 2^k is odd is $k = 3$. We therefore think of our move as removing all of the stones from the third pile and replacing some of the stones so as to leave the position

$$\begin{pmatrix} 25 \\ 28 \\ 5 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

which is an even position.
