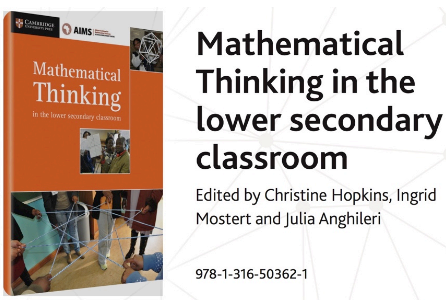
MANAGE YOUR OWN PROFESSIONAL DEVELOPMENT WORKSHOP

These guides are designed to support teachers in developing a deep understanding of the mathematics they teach and in developing more effective ways of teaching.

You can use these guides on your own or as one of a group of teachers who meet together to talk about your mathematics lessons as part of your professional development. Maybe one of you will take the lead in organizing time, date and venue but once you are doing the activities together you will all participate on equal terms in the discussion and reflection.



These Lower Secondary Workshop Guides are chapters in the AIMSSEC Mathematical Thinking Book.

Buy the book online from Amazon or from http://www.cambridge.org/za/education Search for AIMSSEC or for ISBN 9781316503621. To order the book in South Africa go directly to http://www.cup.co.za

For reviews and curriculum map see https://aiminghigh.aimssec.ac.za/mathematical-thinking/

The AIMSSEC App can be downloaded from the internet onto any android smart phone, laptop or tablet in 3 or 4 minutes and it is free.

Go to Google Play and search for **aimssec** and follow instructions. **Please register on the AIMSSEC App so that,** when you are connected to the internet, you can add comments, ask questions and join in professional discussions on the AIMING HIGH Teacher Network website.

After downloading the AIMSSEC App, everything will be available on your own phone or other device so that you will be able to use all the AIMING HIGH resources, lesson activities and professional development workshop guides offline, that is WITHOUT USING THE INTERNET.

From time to time when you connect to the internet you can update your AIMSSEC App content with recently added resources.

EACH WORKSHOP GUIDE HAS A SIMILAR FORMAT:

PAGE 1 **TITLE PAGE**

- Teaching strategy focus. *Each guide focuses on and exemplifies a widelu used teaching methodology.*
- Curriculum content and learning outcomes.
- Summary of mathematical topic (FACT BOX.)
- Resources needed

PAGES 2 & 3 **WORKSHOP ACTIVITIES FOR TEACHERS**

Two pages for you to work through with your colleagues. These activities are to be shared and discussed. For each activity there is a list of resources needed, how to organise the activity (e.g. individual, pairs, whole class), how long the activity will take, when to pause, think and try the activity, and when to record your work.

PAGES 4 & 5 **CLASSROOM ACTIVITIES FOR LEARNERS**

Two pages to help you plan your lesson. You are advised how long to allow for the activity, the resources you might need and the key questions to ask.

PAGES 6 - 10 **CHANGES IN MY CLASSROOM PRACTICE**

Pages on using the teaching strategies with additional resources and activities for use during or after the workshop such as worksheets and templates. For follow-up activities you will find lots more lesson activites on the AIMING HIGH Teacher Network
https://aiminghigh.aimssec.ac.za/category/lesson-activities/

TRANSFORMATION OF FUNCTIONS

Developing systematic working and problem solving skills

by Toni Beardon

Toni has taught for 55 years in schools and universities in the UK, USA and South Africa. At Cambridge University in 1987 she established a new postgraduate teacher training programme and a student community service project. In 1996 she set up the NRICH Online Mathematics Club and other online projects, now the Cambridge University Millennium Mathematics Project. In 2002 Toni founded AIMSSEC. Now retired, Toni divides her time between Cambridge and Cape Town working to provide better educational opportunities for children from disadvantaged communities and to improve the standard of mathematics teaching and learning in Africa in all types of schools.

Transformation of Functions

Teaching strategy: Developing systematic working and problem-solving skills.

Curriculum content: What happens to the graph of $y=f(x)$ when you give values other than 1or 0 to the parameters k, p, a and q in $y = a f(kx + p) + q$?

Prior mathematical knowledge: Be able to draw graphs in four quadrants by point-to-point plotting and be able to recognise graphs of straight lines and parabolas (quadratic functions), and translations and reflections of shapes.

Intended Learning Outcomes At the end of this activity teachers and learners will:

- Recognise translations, reflections and stretches of the graphs of simple functions.
- Understand how changing the parameters in the equation of a function transforms its graph.
- Be able to sketch the graphs of $y=f(kx)$, $y=a f(x)$, $y=f(x+p)$ and $y=f(x)+q$ given the graph of $f(x)$ without using point-to-point plotting .
- Appreciate that what they have learnt about the transformations of linear and quadratic functions also applies to graphs of other functions.
- Have experienced the need to learn for themselves through problem solving rather than being given information by the teacher which they must remember.

Fact box

Transformations of the graph of $y=f(x)$ are given by the graphs of the function $y = af(kx + p) + q$ for different values of the parameters k, p, a and q.

Translation by -p in x direction: $y=f(x)$ maps to $y=f(x + p)$.

Translation by q in y direction: $y=f(x)$ maps to $y=f(x) + q$.

Reflection in x axis: $y=f(x)$ maps to $y= - f(x)$.

Reflection in the y axis: $y=f(x)$ maps to $y=f(-x)$.

Reflection in the line $y =x$: $y=f(x)$ maps to $x=f(y)$.

One way stretch by a factor of a in y direction $y=f(x)$ maps to $y=af(x)$.

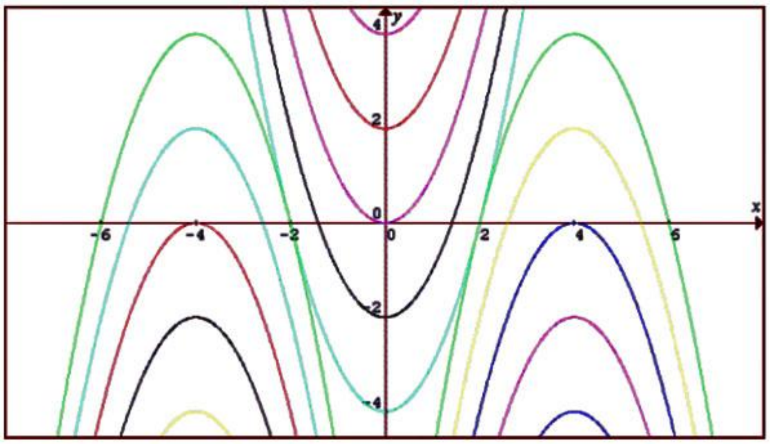
Resources: *One copy of the graphs for each person.*
Computer software such as GeoGebra or a graphic calculator is helpful but not essential.

Workshop Activities for Teachers

Activity 1: Investigation of Parabolic Patterns

Pairs. Every 10 minutes discuss your findings with the whole group. *30 minutes*

The illustration shows part of the graphs of fifteen functions. Two of them have equations:



$y = x^2$ and $y = -(x-4)^2$.

Sketch these two graphs and then find them in this illustration.

All the graphs are transformations of the graph of $y = x^2$.

Use the clues given in this information to help you to find the equations of all the other graphs and to draw the pattern of the 15 graphs for yourself.

Before reading on, stop, try this yourself and discuss it.

Notes to help you do Activity 1

1. Treat this activity like a puzzle to be solved. The shapes of the graphs are all the same but have been moved around, that is they have been **transformed**. You are being asked to sketch a family of graphs. What makes this a family? All the graphs occur by transformations such as reflections and translations of other graphs in the family. The key is to find the simplest function and then to find transformations of the graph of that function that give the other graphs in the family.

2. Notice that the illustration can be treated as three separate sets of five parabolas. Starting with $y = x^2$, the parabola going through the origin, notice that it can be raised or lowered to produce the rest of the parabolas in the middle set by adding or subtracting 2 or 4. This movement parallel to the y-axis is called a *translation* and corresponds to changing the value of q in $y = f(x)+q$.

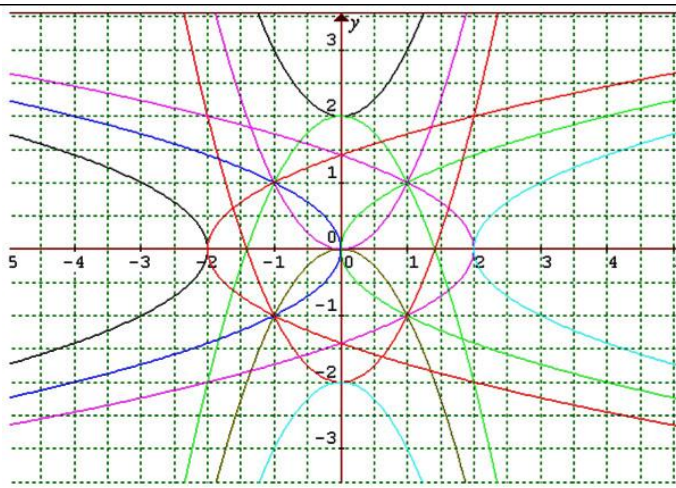
3. Did you discover that $y = -(x-4)^2$ is one of the parabolas in the right hand set? Ask yourself "what is the effect of the minus sign in changing from $y=x^2$ to y= $-x^2$?" and "what is the effect of changing the equation from $y = x^2$ to $y = (x - 4)^2$?"

4. You have probably worked out that the graph of $y= - f(x)$ is a *reflection* of the graph of $y = f(x)$ in the x axis so that the new parabola is an inverted (upside-down) version of the original one. Can you see that the graph of $y = f(x + p)$ is a translation of the graph of $y = f(x)$ by p units to the left if p is positive and by p units in the positive x direction if p is negative? Putting both these ideas together gives you the reason why the graph of $y = -(x-4)^2$ is the parabola y $= x^2$ reflected in the x axis (inverted) and shifted 4 units to the right.

6. Because we are studying families of functions we are interested in what happens as k, p a and q in the equation $y = a\,f(kx + p) + q$ take different values. They are called **parameters because we are interested in what happens as they vary,** whereas, in equations generally, numbers such as k, p, a and q would be called **coefficients**.

The equations for the left hand set of graphs are: $y = -(x + 4)^2$ $y = -(x + 4)^2 + 2$ $y = -(x + 4)^2 + 4$ $y = -(x + 4)^2 - 2$ $y = -(x + 4)^2 - 4$	The equations for the middle set of graphs are: $y = x^2$ $y = x^2 + 2$ $y = x^2 + 4$ $y = x^2 - 2$ $y = x^2 - 4$	The equations for the right hand set of graphs are $y = -(x - 4)^2$ $y = -(x - 4)^2 + 2$ $y = -(x - 4)^2 + 4$ $y = -(x - 4)^2 - 2$ $y = -(x - 4)^2 - 4$

Activity 2: More Parabolic Patterns *30 minutes*



In this illustration there are twelve graphs and the equations of three of them are: $y = x^2$, $x = y^2$ and $x = -y^2+2$

First identify the transformation that occurs when $y = x^2$ maps to $x = y^2$.

Then find which transformation maps $x = y^2$ to $x = -y^2$ and finally to $x = -y^2 + 2$.

Can you find the equations of the other nine graphs?

. Before reading on, try this and discuss it.

More Activities: Try the activities given for the classroom on the next page.

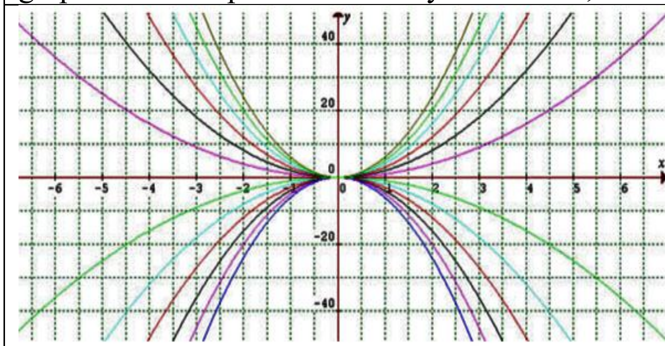
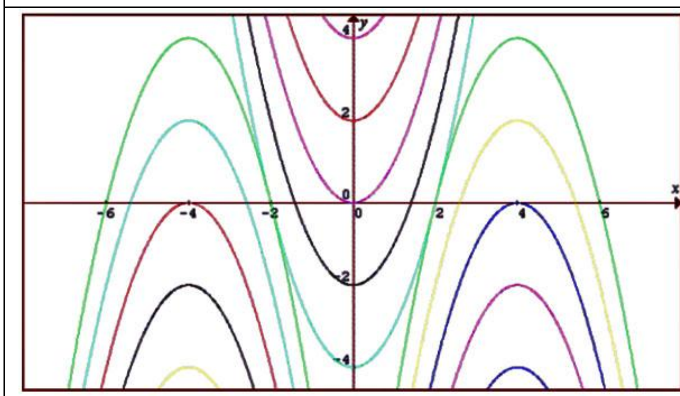
Teaching issues to discuss

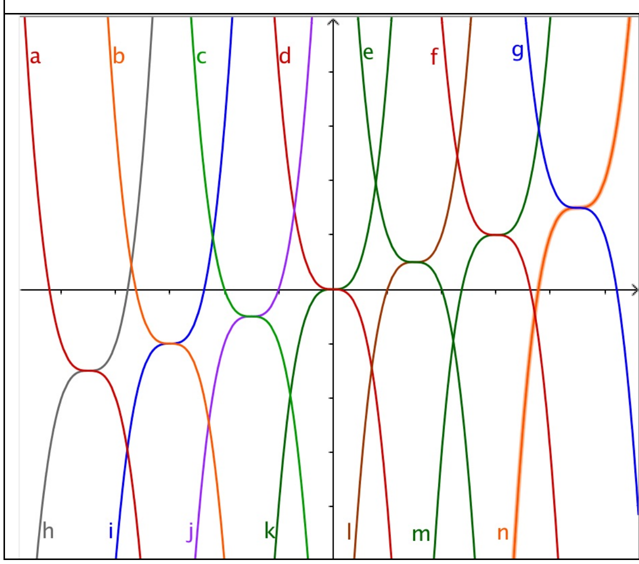
1. Why teach this topic? If you need to work with a complicated function, mathematicians transform it to the simplest function in the family, work with the simple function (which is easier), then transform your answer back to the complicated function you started with. This technique is often used in higher mathematics.
2. A graphing package can be used for this and similar problems. The point of the activity is not to practice plotting graphs by hand but to learn the general features of the changes in graphs when the function is transformed by altering the parameters *k, p, a* and *q* in the equation: $y = a\,f(kx + p) + q$.
3. The benefit of using a graphing package is that it enables the learner to experiment by changing the parameters and observing the effect on the graph. If the learners make mistakes they can see the actual shape and position of the graphs they have drawn and learn from their mistakes.
4. This type of problem is sometimes called an *inverse* problem, The exercise requires working backwards from what is most often given as the answer to a question to find out what question would have led to that answer. This is a particularly useful type of problem because it calls for mathematical thinking and creativity and not just remembering facts learnt by rote.

Classroom Activities for Learners

Activity 1: Families of Graphs and Transformations

Computer software such as GeoGebra or a graphic calculator is helpful but not essential.
Work in pairs. Every 15 minutes discuss your findings with the whole group. *1½ hours*

<table>
<tr><td colspan="2">In each of the following illustrations of families of graphs the equations of some graphs are given and you are asked to find the equations of the other graphs. All the graphs are transformations of the graph of the simplest function by translation, reflection or one way stretch.</td></tr>
<tr><td>
</td><td>1.In this illustration there are twelve graphs and three of the equations are:
$y = x^2$
$y = 2x^2$
$y = - x^2$.
Sketch these three graphs and then locate them in the illustration. Find the equations of the other nine graphs. What transformations map the graph of $y = x^2$ to the other graphs?</td></tr>
<tr><td>
</td><td>2. The illustration shows part of the graphs of fifteen functions. Two have equations:
$y = x^2$
$y =-(x - 4)^2$
Sketch these two graphs and then locate them in this illustration. Use the clues given in this information to help you to find the equations of all the other graphs and to draw the pattern of the 15 graphs for yourself.

What transformations map the graph of $y = x^2$ to the other graphs?</td></tr>
<tr><td>
</td><td>3. This is a family of cubics. Four of the equations are:
$y = x^3$
$y = -x^3$
$y = (x + 6)^3 - 2$
$y = -(x - 9)^3 + 3$
Can you locate these four graphs and find the equations of the other ten graphs?
What transformations map the graph of $y = x^3$ to the other graphs?</td></tr>
</table>

Activity 2 Solving Quadratic Equations and Transforming Parabolas

Work in pairs *40 minutes*

Consider the quadratic equation: $2x^2 + 9x - 3 = 0$.
Replace the **?** marks in the working below to complete the steps in solving this equation. Work with a partner. Talk to each other about the best way to explain each step. This method is called 'completing the square', why do you think it has this name? The equation can be rearranged to give the following forms:

$2[x^2 + \frac{9x}{2} - \frac{3}{2}] = 0.$

$$2\left[(x + ?)^2 - \frac{3}{2} - \frac{81}{16}\right] = 0$$

$$2[(x+?)^2 - ?] = 0$$

$$(x+?)^2 = ?$$

So the solutions are:

$x = -\frac{9}{4} \pm \frac{\sqrt{105}}{4} = \frac{-9 \pm 10.247}{4}$ = **?** and **?** (to 3 decimal places)

Deriving the quadratic formula	
To solve the quadratic equation $ax^2 + bx + c = 0$ we rearrange it to complete the square. Replace the **?** marks below.	Think about where the graph of $y = ax^2 + bx + c$ cuts the y axis.
$a(x+?)^2 + c - \frac{b^2}{4a} = 0$	Rearranging the equation.
$a(x + ?)^2 - \frac{b^2 - ?}{4a} = 0$	The graph is a translation of $y=x^2$ by $-b/2a$ in the x direction and $\frac{b^2-4ac}{4a}$ in the negative y direction so it is symmetrical about the line $x = ?$
$(x + ?)^2 = \frac{b^2 - ?}{4a^2}$	
Next take the square roots of both sides $(x+?) = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$	The roots of the equation are given by the x coordinates of the points P and Q which are symmetric on either side of the line $x =$ **?**
The roots of the equation are: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	P = (**?**) Q = (**?**)

Notes on Activity 2

Deriving the quadratic formula

To solve the quadratic equation $ax^2+bx+c=0$ we rearrange it to complete the square.	Think about where the graph of $y=ax^2+bx+c$ cuts the y axis.
$a(x+\frac{b}{2a})^2+c-\frac{b^2}{4a}=0$	Rearranging the equation.
$a(x+\frac{b}{2a})^2-\frac{b^2-4ac}{4a}=0$	The graph is a translation of $y=x^2$ by $-b/2a$ in the x direction and $\frac{b^2-4ac}{4a}$ in the negative y direction so it is symmetrical about the line $x=-b/2a$
$(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$	
Next take the square roots of both sides $(x+\frac{b}{2a})=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$	The roots of the equation are given by the x coordinates of the points P and Q which are symmetric on either side of the line $x=-b/2a$
This gives the formula for the solutions of the equation: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	$P=(\frac{-b-\sqrt{b^2-4ac}}{2a},0)$ $Q=(\frac{-b+\sqrt{b^2-4ac}}{2a},0)$

<u>The solution step by step</u>

To solve $2x^2+9x-3=0$ by completing the square, the equation can be rearranged to give:

$$2\left[\left(x+\frac{9}{4}\right)^2-\frac{3}{2}-\frac{81}{16}\right]=0$$

$$2\left[\left(x+\frac{9}{4}\right)^2-\frac{105}{16}\right]=0$$

$$\left(x+\frac{9}{4}\right)^2=\frac{105}{16}$$

So the solutions are:

$$x=-\frac{9}{4}\pm\frac{\sqrt{105}}{4}=\frac{-9\pm10.247}{4}$$

$= -2.25 \pm 2.562$

$= -4.812$ and 0.312 (to 3 d.p.)

Repeat this task to sketch the graph of $y=3x^2+5x-1$ and to show the solutions of the equation $3x^2+5x-1=0$.

Changes to my classroom practice

1. The problems are presented for learners themselves to work on with a minimal amount of help from the teacher. However, teachers may like to use some of the problems as the basis for class discussion. Instead of giving the equations of some functions and asking learners to sketch the graphs, this challenge gives the graphs and asks learners to find their equations. This encourages learners to experiment by changing the equations *systematically* to discover the effect on the graphs.

2. The type of task is sometimes called an inverse problem. Rather than being given a question and finding the answer, in an inverse problem the 'answer' is given (in this case the set of graphs) and the learner has to find the 'question' (in this case the equations of the graphs). Because each set of graphs forms a pattern the equations can be found by seeing the pattern and by working systematically to find the transformations of the simple graph that produce the other graphs in the set. Equations can be checked by trying particular helpful points, e.g. when $x = 0$.

3. It is suggested that learners experiment with sketching the graphs until they have found the equations for all the graphs matching those in the given picture. They will need to identify the graphs in the illustration that correspond to the given equations, then to experiment to find how they should change the equations to produce translations, reflections and stretches of the graphs. When they have found the equations for all the graphs they should explain the reasoning by which they have found the appropriate functions and their equations. The use of graphing software is very helpful in this investigation but not essential.

4. Through playing and experimenting with these patterns, and using trial and improvement methods, learners will develop graph sketching skills and knowledge and understanding of transformations of graphs and the corresponding equations of the graphs in each family. Then they can be creative in designing and making their own graph patterns.

5. Actually this topic, namely the recognition of compositions of transformations of graphs (reflections, translations and stretches) and the relationship between the graphs of $y=f(x)$ and $y=a\,f(kx + p)+ q$, where a, k, p and q are parameters, usually occurs in the mathematics curriculum in most countries in the last year two years in school. Graphing software opens up the possibility of learning this topic more easily, and at a younger age. It also gives us a better capacity for visualisation and a more intuitive feeling for, and understanding of, the interaction of the geometry and the algebra.

6. As a possible approach you could start the lesson by showing the picture in question 1 and asking learners to identify the graphs of $y=x^2$ and $y = -x^2$ and $y=-(x-4)^2$. Encourage discussion about the similarities and differences between these graphs, and their equations.

7. Give learners time to experiment with the aim of finding rules for changing equations to transform graphs. Working in pairs helps learners to work out their ideas through discussion.

8. While the learners are working to find the equations the teacher should observe what they are doing and listen to what they say. If they need help, try to ask questions to focus the learners' attention on the important aspects of the problem and try not to tell them what to do. The aim is that they should find the equations for themselves, perhaps helping each other. By discussing the patterns they will develop their own understanding. Observing the learners at this stage will

help the teacher to in decide how to lead the class discussion to follow.

9. The teacher should lead a class discussion only when the learners have had time to find the equations. The teacher should ask the learners to demonstrate and explain what they have done and then draw the ideas together with the purpose of defining a set of rules that can be applied in other similar situations.
10. Discuss the equations and the transformations in question 1 before moving on to questions 2 and 3. One of these can be given for homework.
11. As another problem for the learners, you can use the family of graphs from Activity 2 in the Teacher Workshop in the same way.
12. A nice extension is for pairs to create a picture of their own, which they then challenge another pair to reproduce. An attractive wall display of learners' work could be made from this.
13. It is important for learners to be able to learn independently. When you introduce Classroom Activity 2 about quadratic equations you should explain to the learners at the start of the lesson that they must read all the information for themselves and answer the question, and you are not going to tell them how to do because you want them to think for themselves. You may want them to work by themselves for about 5 to 10 minutes and then to work with a partner. Much later in the lesson you should conduct a class discussion in which you base your explanations on what the learners have achieved and what they say about it.

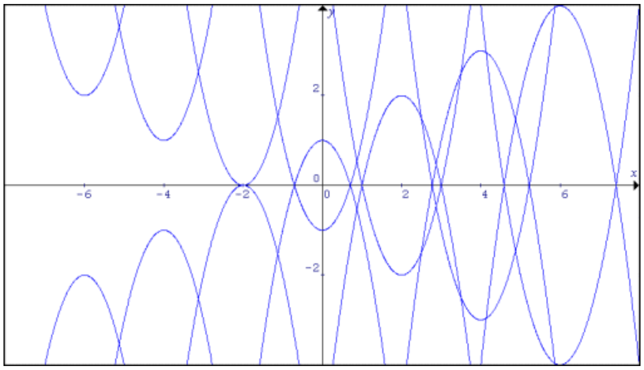
    Classroom Activity 2 is designed for learners to investigate for themselves the connection between the graphs $y = x^2$ and $y = ax^2 + bx + c$ and how this illustrates the solutions of a quadratic equation. The formula for the solutions is derived by completing the square in $ax^2 + bx + c = 0$ rewriting the equation as $(x + b/2a)^2 = (b^2 - 4ac)/ 4a^2$ and then taking square roots.

    For the learners, Classroom Activity 2 is an exercise in sketching the graph for one example, identifying the translations from the simple graph $y=x^2$ and explaining how this connects with the solution of the equation by completing the square and the quadratic formula. You may choose to do this activity at a different time from Activity 1 but it is helpful for learners to make the connection between the geometry and the algebra (deriving the quadratic formula and solving equations).

Key questions to develop understanding

- You are being asked to sketch a family of graphs. What makes this a family?
- Can you pick out the simplest member of the family? It will usually go through the origin.
- Comparing the equation of this simplest function with the general equation $y = af(kx + p) + q$ what are the values of the *parameters k, p, a and q*?
- What happens when you change these coefficients one by one?
- What is the same and what is different about the equations of those two graphs (for example $y=x^2$ and $y=-(x-4)^2$)?
- How might these similarities and differences relate to the way the graphs look and their positions on the axes?
- Can you convince us that the rules you have found will work with graphs of other functions?

Questions to check your knowledge and understanding



1.The equations of three of the graphs are:
$y = 2(x-6)^2 - 4$
$y = 2x^2 - 1$
$y = -2(x-4)^2 + 3$

What is the same and what is different about the equations $y=x^2$ and $y = 2(x-6)^2 - 4$?
How might these similarities and differences relate to the way the graphs look and to their positions on the axes?

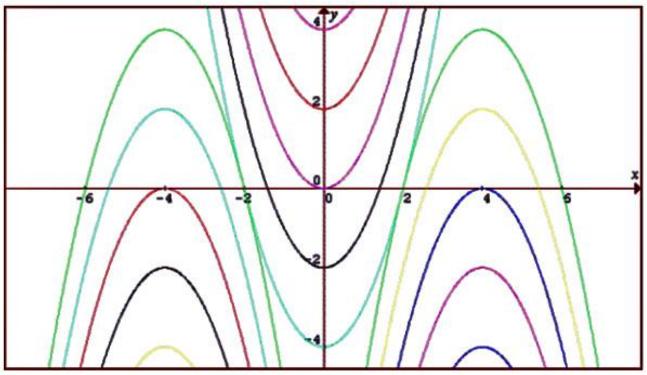
Find the equations of the other 11 graphs in this pattern. What makes it a family?

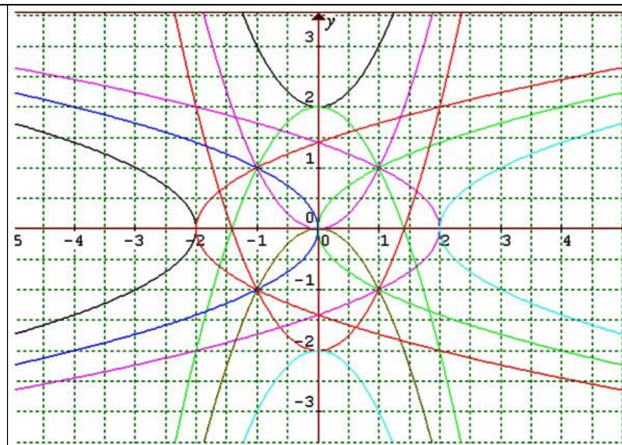
Make your own sketch of this family of graphs and label each function with a single letter, then make a list of all the corresponding equations.

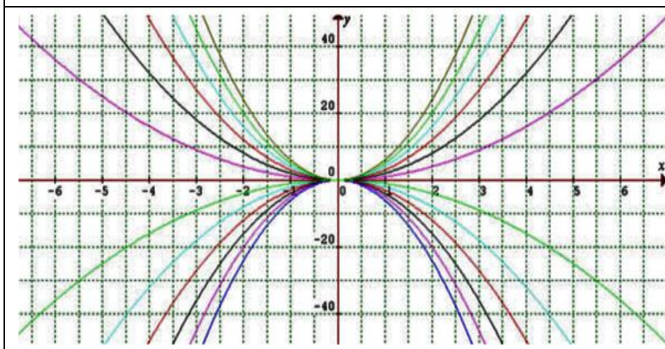
2. Explain why the rules discussed in this chapter will work with graphs of other functions.

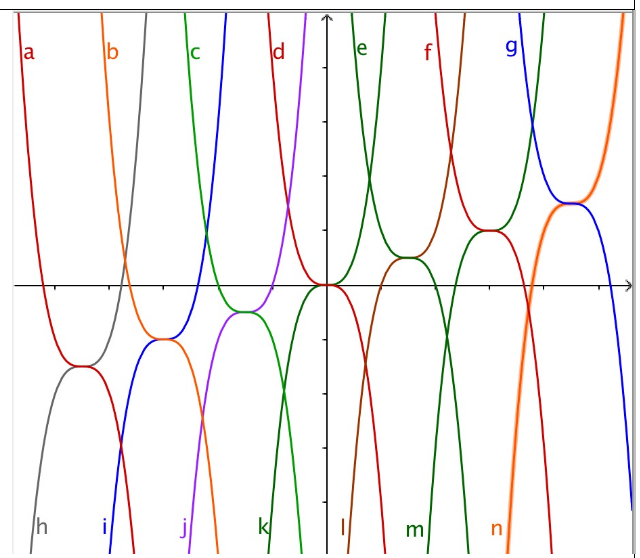
3. Create your own pattern of graphs of a family of functions. You could exchange your pattern with a partner and challenge them to find the equations.

 See http://nrich.maths.org/10774 for a video showing the transformations of the graph of $y=x^2$.





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