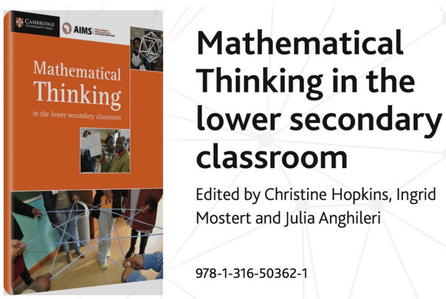
WORKHOP GUIDES FOR TEACHERS TO LEARN TOGETHER
UPPER SECONDARY GT7 APPLICATIONS OF TRIGONOMETRY
Guide for your own self-help professional development workshop
and resources for inquiry based lessons.

MANAGE YOUR OWN PROFESSIONAL DEVELOPMENT WORKSHOP

These guides are designed to support teachers in developing a deep understanding of the mathematics they teach and in developing more effective ways of teaching.

You can use these guides on your own or as one of a group of teachers who meet together to talk about your mathematics lessons as part of your professional development. Maybe one of you will take the lead in organizing time, date and venue but once you are doing the activities together you will all participate on equal terms in the discussion and reflection.



These Lower Secondary Workshop Guides are chapters in the AIMSSEC Mathematical Thinking Book.

Buy the book online from Amazon or from http://www.cambridge.org/za/education Search for AIMSSEC or for ISBN 9781316503621. To order the book in South Africa go directly to http://www.cup.co.za

For reviews and curriculum map see https://aiminghigh.aimssec.ac.za/mathematical-thinking/

The AIMSSEC App can be downloaded from the internet onto any android smart phone, laptop or tablet in 3 or 4 minutes and it is free.

Go to Google Play and search for **aimssec** and follow instructions. **Please register on the AIMSSEC App so that,** when you are connected to the internet, you can add comments, ask questions and join in professional discussions on the AIMING HIGH Teacher Network website.

After downloading the AIMSSEC App, everything will be available on your own phone or other device so that you will be able to use all the AIMING HIGH resources, lesson activities and professional development workshop guides offline, that is WITHOUT USING THE INTERNET.

From time to time when you connect to the internet you can update your AIMSSEC App content with recently added resources.

EACH WORKSHOP GUIDE HAS A SIMILAR FORMAT:

PAGE 1 **TITLE PAGE**

- Teaching strategy focus. *Each guide focuses on and exemplifies a widelu used teaching methodology.*
- Curriculum content and learning outcomes.
- Summary of mathematical topic (FACT BOX.)
- Resources needed

PAGES 2 & 3 **WORKSHOP ACTIVITIES FOR TEACHERS**

Two pages for you to work through with your colleagues. These activities are to be shared and discussed. For each activity there is a list of resources needed, how to organise the activity (e.g. individual, pairs, whole class), how long the activity will take, when to pause, think and try the activity, and when to record your work.

PAGES 4 & 5 **CLASSROOM ACTIVITIES FOR LEARNERS**

Two pages to help you plan your lesson. You are advised how long to allow for the activity, the resources you might need and the key questions to ask.

PAGES 6 - 10 **CHANGES IN MY CLASSROOM PRACTICE**

Pages on using the teaching strategies with additional resources and activities for use during or after the workshop such as worksheets and templates. For follow-up activities you will find lots more lesson activites on the AIMING HIGH Teacher Network
https://aiminghigh.aimssec.ac.za/category/lesson-activities/

Applications of Trigonometry

Mathematical modelling and making real life connections

By Jenny Orton

Jenny taught in High Schools in East Anglia, and worked as a mathematics advisor in various parts of England. She worked with Guinean and European colleagues on a teacher development project in Guinea Bissau for 18 months in 2004/5. Jenny has also worked as an Education Technology Consultant, working with teachers across the UK, in Eire and even in Saudi Arabia (very briefly!). She is currently Principal Examiner for Further Pure Mathematics with Cambridge Assessment International Education (better known as CIE). Beyond Mathematics (is there such a place?) she enjoys caring for her 92 year old father, and babysitting grandchildren, preferably not more than four at a time.

Applications of Trigonometry

Teaching Strategy: Mathematical modelling and making real life connections

Curriculum content: Applying trigonometry to solving real life problems

Prior knowledge needed: Understanding of the sine, cosine and tangent ratios in right-angled triangles, and familiarity with the sine and cosine rules for other triangles.

Intended Learning Outcomes At the end of this activity teachers and learners will:

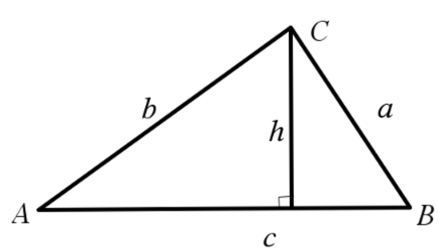
- Know how to solve problems involving missing sides and angles in triangles
- Understand when to apply the sine and cosine rule rather than sine, cosine or tangent ratios
- Be able to model a situation by sketching the appropriate diagram
- Appreciate which ratio or rule is appropriate to a given situation
- Have experienced practical work with a clinometer.

Fact box

The sine and cosine of an angle are uniquely defined as the lengths of edges in a right angled triangle with hypotenuse 1 unit and the tangent is defined to be sine divided by cosine. It follows that, in any right angled triangle the three basic ratios can then be used to find missing lengths or angles :-

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \text{and } \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

The Sine and Cosine Rules, used to find missing lengths or angles in any triangle, are:-



Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin c}$$

Cosine Rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

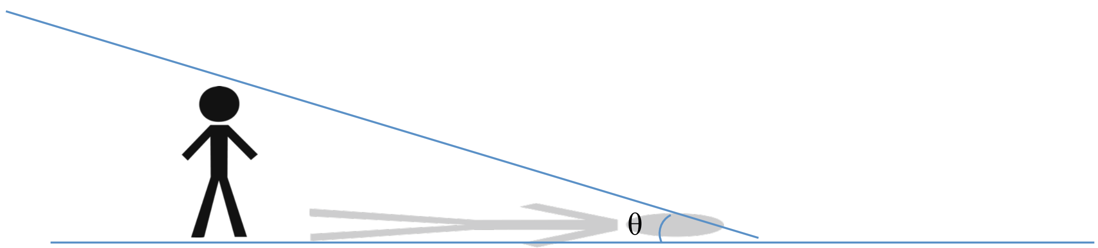
Resources needed: Protractors, drinking straws, sticky tape, string and weights to make clinometers, tape measures or metre rules or newspaper to make measuring sticks (see page 6).

Workshop Activities for Teachers

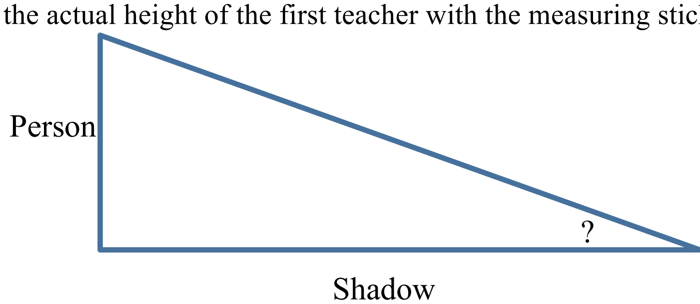
Activity 1: Me and my shadow – angle of elevation

Measuring stick *Pairs, whole group* *30 minutes*

This is an activity that you can use with your learners too. Very little equipment is needed, and only a short time collecting data outside. Do warn them never to look directly at the sun though! The aim is to work out the angle of elevation of the sun by measuring the length of a person's shadow. This changes throughout the day, so try to repeat the activity at different times of day. You could then look at the results on a graph, plotting time on the horizontal axis, angle of elevation on the vertical axis.



You do need a sunny day for this!! In pairs, take a measuring stick into an empty space outside. One teacher stands with their back to the sun whilst the other measures the length of their shadow, and checks the actual height of the first teacher with the measuring stick.



Back inside, sketch the right angled triangle formed by the person, their shadow and the beam of sunlight which goes just over their head, defining the very end of the shadow. The angle of elevation of the sun is then the angle marked with a question mark in the diagram.

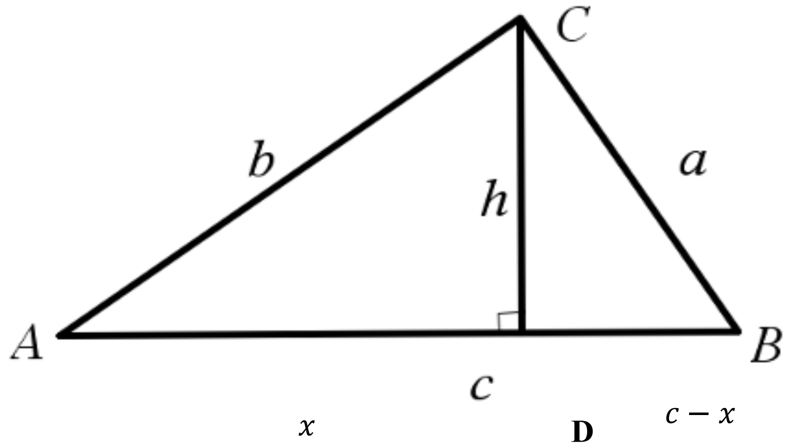
Which trig ratio is going to be used to find the angle of elevation of the sun? Use this ratio to work out the angle. You might like to work out the length of the ray of the sun too – how many different ways can you find to do this?

Notes:

1. As you know the opposite and adjacent sides, it is the tangent ratio required! The most direct way to work out the length of the ray (the hypotenuse of the triangle) is using Pythagoras rule, but sine or cosine can also be used once the angle is known.
2. If you have time you might look at the suggested Classroom Activities for Learners and see how they can use a clinometer to measure angles of elevation. You might discuss how you could use these activities with the learners you teach.
3. You might want to discuss how younger learners might carry out this task, and also the Classroom Activities on pages 6 and 7, without using trigonometry, by taking the same measurements and then making accurate scale drawings.

Activity 2: Making statements

Copy of diagram *Individual, pairs* *30 minutes*



This exercise will help revise not only the trigonometrical ratios in right-angled triangles, but also develop steps towards proving the Sine Rule and the Cosine Rule. It also highlights the standard labelling for triangles with capital letters for the vertices, and lower case letters for the corresponding opposite sides.

Individually – write down as many true statements as you can about the diagram. You can think from the point of view of your learners, and make some very simple statements, and then develop some statements about the relationships between the sides and angles. You might want to differentiate by setting different targets for stronger learners if you use the activity in class.

In pairs – compare your statements with your partner, and make sure that you can justify them. Pool your ideas.

As a whole group make sure that you have included statements using the trigonometry of right angled triangles.

In groups of 3 or 4, try to develop the proof of the sine rule and cosine rule using the statements made.

What can you deduce from the cosine formula if angle C is a right angle?

Notes:

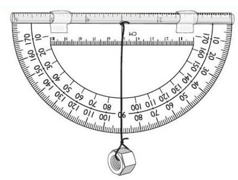
1. Examples of possible statements: sin A = h/b, $b$sinA = h, cosB = $(c - x)/a$.
2. The sine rule can be shown by using $\boldsymbol{h = b \sin A}$ from one right angled triangle, $\boldsymbol{h = a \sin B}$ from the other and then $\boldsymbol{b \sin A = a \sin B}$.
3. To prove the Cosine Rule, use $\boldsymbol{x = b \cos A}$ and then find two expressions for $\boldsymbol{h^2}$ – one from each right angled triangle.
4. If angle C is a right angle then cos C = 0 and $c^2 = a^2 + b^2 - 2ab \cos C$ gives $a^2 + b^2 = c^2$. Note that this is the converse of Pythagoras Theorem and not Pythagoras Theorem as such. Also note that Pythagoras Theorem is a special case of the cosine rule which in turn is a generalisation of Pythagoras Theorem.

Classroom Activities for Learners

Activity 1: Finding heights

Drinking straw, sticky tape, protractor, string, weight, tape measure, metre rule or newspaper to make measuring sticks.

Groups of three *45 minutes*

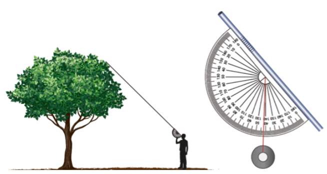


This practical activity can be used to put trigonometry into context, firstly using right angled triangles. The construction of a clinometer and a measuring stick is simple and doesn't take long, so the focus of the lesson is on taking measurements and solving the problem.

As the picture shows, a clinometer consists of a drinking straw sight stuck to the straight edge of a protractor, with a weight tied on a string fixed through the midpoint of this side.

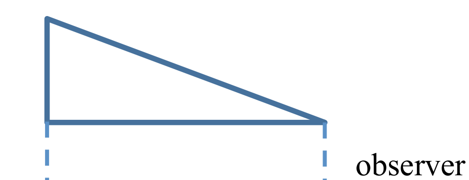
Metre measuring sticks can be made with tightly rolled newspaper and then accurately marked into 10 cm and 1 cm lengths using a shorter ruler.

Group the students in threes and get them to construct a clinometer and a metre stick.



Once the learners have made their clinometers, they can go out and chose a tree or building to estimate its height. One learner stands a certain distance away from the object, and takes a sight of the top of the object through the drinking straw. The second learner reads off the angle that the string makes with the vertical – that will also be the angle that the straw makes with the horizontal – see if the learners can explain why. The third learner measures the distance between the person holding the clinometer and the foot of the object being measured, as well as making a note of the height of the clinometer above the ground.

Back in the classroom, get the learners to draw a diagram of the situation, not forgetting to include the height of the observer. They should mark on the known distance and the known angle. Get them to decide which trigonometric ratio to use, and then calculate the height of the object, not forgetting to add in the height of the clinometer.



Ideas for Teaching Activity 1:

Get different groups to find the height of the same object from different positions and get them to discuss why their answers are different, which brings questions about appropriate degrees of accuracy. Always get learners to check if their answers are of the right order of magnitude.

Activity 2: Finding Heights 2

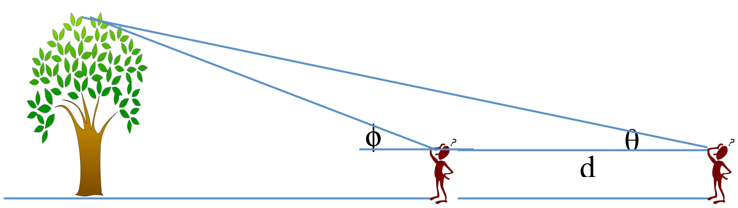
Measuring stick, clinometers *Threes* *30 minutes*

In this practical work, learners need the clinometers that they constructed for the previous activity.

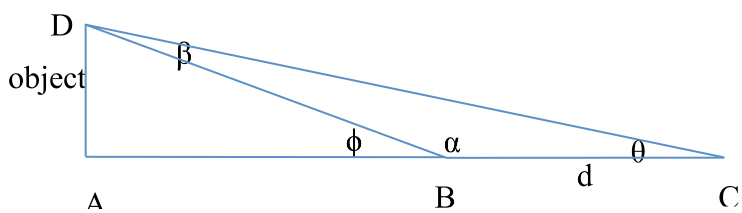
Discussion point in classroom: What if the object that you want to measure is the other side of a river? Can you still calculate its height using the clinometer?

In this case learners can take two measurements using the clinometer from different distances.

Organise the learners in groups of three, with two clinometers and a measuring stick for each group. Take them outside and decide what they are going to find the height of – it's a good idea to have two groups working on each object so that they can compare results.
The two learners with clinometers line up and the third learner reads the two angles of inclination (ϕ and θ) and measures the distance between them (d) and make a note of the height of the clinometer above the ground. They could do this more than once, and then check the accuracy of their final answers, allowing for a discussion of where the inaccuracies might arise.



Back in the classroom, learners can draw their diagram with the key measurements (d, θ and ϕ) working out the angles α and β.



Since they know the pair d and β, it is possible to use the Sine Rule to calculate the length of BD (or CD).

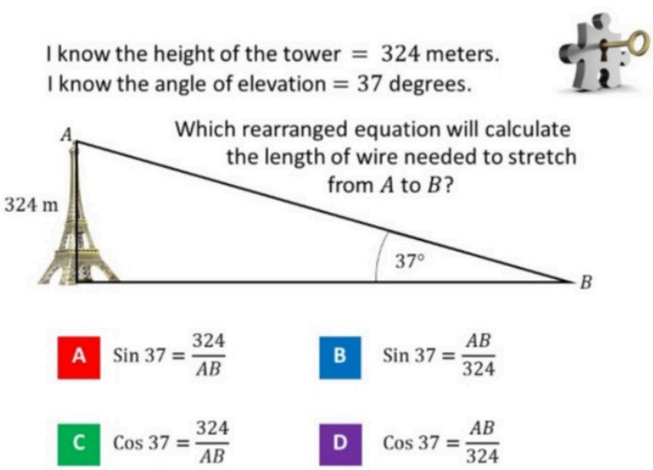
From this they can calculate the length of AD. Remind them to add on the height of the clinometer above the ground to their answer.
See the example on Page 8 for help with the calculations.

Ideas for Teaching Activity 2

As a check, another way that your learners might estimate the height of a building (for example the school building) would be to measure the height of one course of bricks plus mortar, and count the number of courses of bricks. If your learners visit a town or city with tall buildings it would be interesting for them to look at the buildings and to estimate the heights by counting the number of floors and thinking about how tall each floor must be.

Checking knowledge

1. Before modelling with trigonometry can take place, learners do need to be secure in their knowledge of the techniques to be applied. A review of these should take place before embarking on problem solving. One way of making this revision interesting would be to get learners to design a poster with all the key facts on them. This should include relevant diagrams which will reinforce the important part that visual representation plays in this problem solving. With more able groups, you could use the diagram from the second Teachers' Activity, and not only review the ratios, but also develop the proofs for Sine and Cosine Rules.
2. Formative assessment using diagnostic questions can be very quick and effective. See https://diagnosticquestions.com/Questions Here is an example copied from this site (with permission). This should take about 5–10 minutes.



a. Write the question on the board, say to the class:

b. "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".

c. Notice how the learners respond. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.

d. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.

e. Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers. It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.

f. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

3. A more open-ended way of getting learners to practice all their trigonometry knowledge is to challenge them to find as many triangles as they can which obey exactly 3 of the following rules:-
   i. One side is 3cm   ii. One angle is 90^o iii. One side is 4cm   iv. One angle is 30^o.

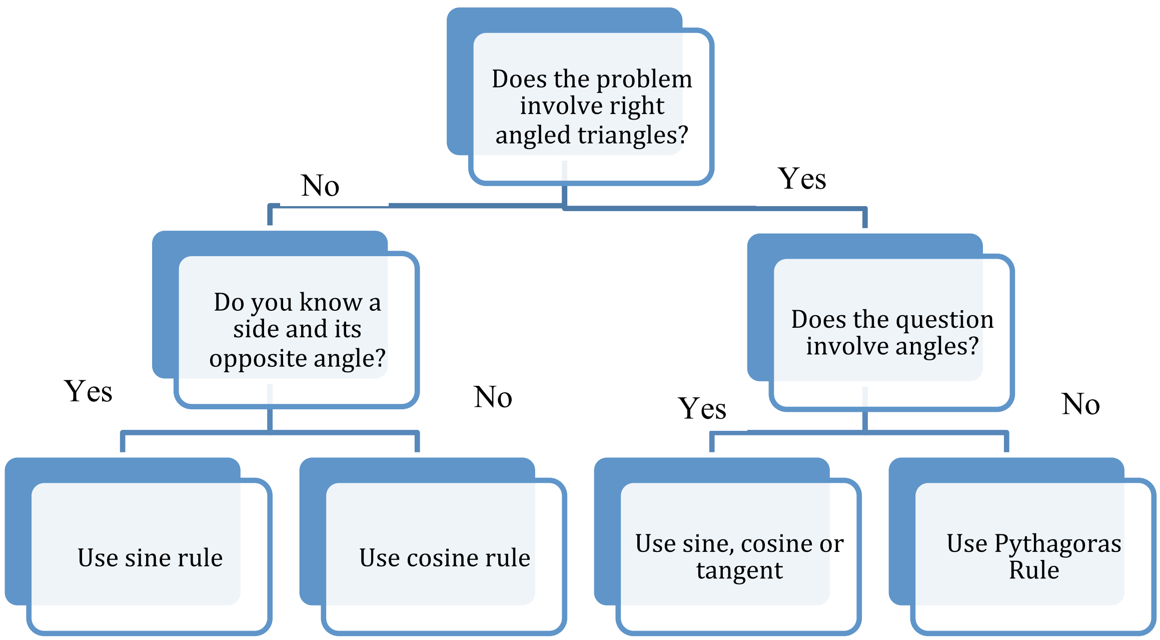
For each of their triangles, learners should work out the missing side(s) and angle(s) using the relevant trigonometry or Pythagoras.

This problem comes from a website called RISPs http://www.risps.co.uk/ and the answer (12 triangles in total) can be found there – it's problem 24!

Common Errors and Misconceptions

Do make sure you make learners aware of the ambiguous case that can arise when using the Sine Rule as there are always two values between 0^o and 180^o when finding the inverse sine. In practical contexts it is usually obvious which value to choose.

Problem Solving Techniques:
Do remind learners that equilateral triangles can be split into two right angled triangles so they should know the trig ratios of the special angles 30° and 60° in terms if 1, 2 and √3, and also the trig ratios for 45° from the right angled isosceles triangle. Remind them also that other isosceles triangles can be split into two right angled triangles and only Pythagoras, sine, cosine or tangent ratios are needed. It's a good idea to show a simple flow chart of the decisions to be made when solving a problem involving trigonometry – you or the learners could make this into a poster.



A key strategy used in this chapter is representing the real life situations that are to be modelled by using sketches, filling in the information known, and identifying exactly what they are trying to find out. Making the connection between the situation and the mathematics needed to find missing information is often helped by a visual representation, so emphasize the need for a diagram at every stage.

Key Questions
One of the most important features of modelling and solving problems is to make sure that your answer is of the right order of magnitude, so a key question to be asked is **'Is your answer reasonable?'** Get learner to discuss whether their answer should be bigger or smaller than any given measurements, giving reasons why, as well as making sure their answer is reasonable in the context of the problem. A useful hint is that the side opposite the biggest angle in a triangle should be the longest. This is particularly helpful when applying the Sine Rule to find a missing angle where two answers are possible.
As learners embark on mixed problems with trigonometry, they need to be able to make decisions about which technique is relevant to a particular situation, so prompting them regularly by asking them '**What do you know, what have you got to find out'** will help them focus on analyzing the situation in order to go through the flow chart shown above.

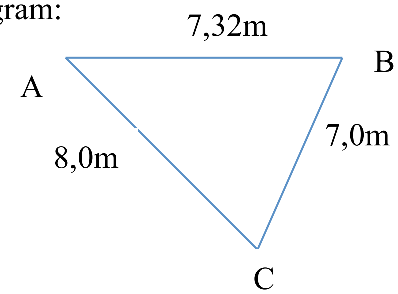
Follow up activities:

There are many real life contexts and questions that can be posed involving applying trigonometry. Try to choose some that will interest your learners, and some which have social relevance. Here are some starting points, but you and your class will be able to think of others!

1) **Football:**

The goal posts in football are 7,32 metres apart. A ball is placed on the ground 8 metres from one goal post, and 7 metres from the other. Within what angle must the ball be kicked along the ground to score a goal? (*Note the use of a decimal comma rather than a decimal point. Some countries use one, some the other so just use the one you are accustomed to.*)

Worked answer: Start with a diagram:



Angle C is the angle required!

This is not an isosceles or equilateral triangle and following through the flowchart, we can see that the Cosine Rule is needed with angle C.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{7{,}0^2 + 8{,}0^2 - 7.32^2}{2 \times 7{,}0 \times 8{,}0} = 0.5305$$

$$\text{so } C = \cos^{-1} 0{,}5305 = 57{,}96° = 58°$$

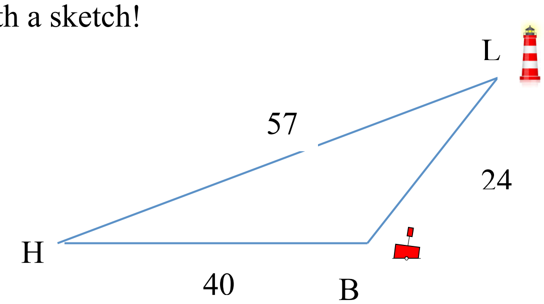
Discuss appropriate rounding in the context of kicking a football!

You can change the distances of the ball from the goal posts to create a worksheet, or change the sport to give a different distance between the posts.

2) **Compass bearings**

A small boat leaves a harbour (H) and travels due East for 40km to a marker buoy (B). At B the boat turns left and sails for 24km towards a lighthouse (L). It then returns to harbour, a distance of 57km. Find the bearing of the lighthouse from the harbour.

Worked solution: Start with a sketch!



We need to find angle H.

Follow the flow chart through to choose the right method – cosine rule again!

$$h^2 = l^2 + b^2 - 2lb \cos \mathrm{H}$$

$$\cos\mathrm{H} = \frac{l^2 + b^2 - h^2}{2lb} = \frac{40^2 + 57^2 - 24^2}{2 \times 40 \times 57} = 0{,}9379$$

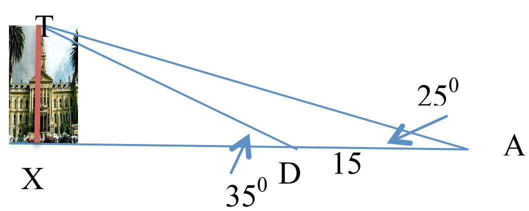
$$\text{so C} = \cos^{-1} 0{,}9379 = 20.4°$$

The bearing of the Lighthouse from H is therefore 90 – 20.4° = 070° to the nearest degree.

3) **Return to the clinometer!**

The angle of elevation of the top of a building measured from point A is 25°. At point D which is 15m closer to the building, the angle of elevation is 35° Calculate the height of the building.

Worked solution:



Angle T = 180 – 25 - 145 = 10^0 and we have a known pair of angle and opposite side, so we use the Sine Rule to find the length TD:-

$$\frac{TD}{\sin 25} = \frac{15}{\sin 10} \quad so\ TD = \frac{15 sin25}{\sin 10} = 36.5$$

And then we have a right angled triangle TDX and can find TX:- TX = 36.5 x sin35 = 20.9 m

Disabled Ramps

There are strict regulations governing the slope on disabled ramps and exploring these can make another interesting project for students. (https://law.resource.org/pub/za/ibr/za.sans.10400.s.2011.html)

If the maximum steepness allowed is 1:12 , i.e. a vertical rise of 1 unit for every 12 units horizontally, what would this mean in terms of the angle of elevation of a ramp? Maybe there are some steps in your school where a disabled person would need a ramp. What would the length of the ramp need to be? Can your learners design one?

Roof Trusses You might be able to spot some of these in your school building and build a worksheet around them!