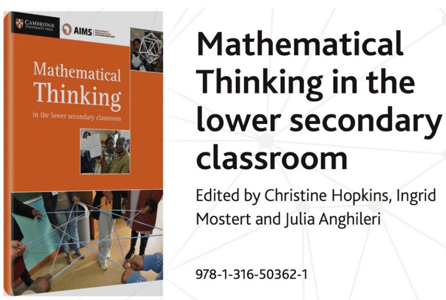
WORKHOP GUIDES FOR TEACHERS TO LEARN TOGETHER
UPPER SECONDARY: GT6 Trigonometric identities and equations
Guide for your own self-help professional development workshop
and resources for inquiry based lessons.

MANAGE YOUR OWN PROFESSIONAL DEVELOPMENT WORKSHOP

These guides are designed to support teachers in developing a deep understanding of the mathematics they teach and in developing more effective ways of teaching.

You can use these guides on your own or as one of a group of teachers who meet together to talk about your mathematics lessons as part of your professional development. Maybe one of you will take the lead in organizing time, date and venue but once you are doing the activities together you will all participate on equal terms in the discussion and reflection.



These Lower Secondary Workshop Guides are chapters in the AIMSSEC Mathematical Thinking Book.

Buy the book online from Amazon or from http://www.cambridge.org/za/education Search for AIMSSEC or for ISBN 9781316503621. To order the book in South Africa go directly to http://www.cup.co.za

For reviews and curriculum map see https://aiminghigh.aimssec.ac.za/mathematical-thinking/

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Go to Google Play and search for **aimssec** and follow instructions. **Please register on the AIMSSEC App so that,** when you are connected to the internet, you can add comments, ask questions and join in professional discussions on the AIMING HIGH Teacher Network website.

After downloading the AIMSSEC App, everything will be available on your own phone or other device so that you will be able to use all the AIMING HIGH resources, lesson activities and professional development workshop guides offline, that is WITHOUT USING THE INTERNET.

From time to time when you connect to the internet you can update your AIMSSEC App content with recently added resources.

EACH WORKSHOP GUIDE HAS A SIMILAR FORMAT:

PAGE 1 **TITLE PAGE**

- Teaching strategy focus. *Each guide focuses on and exemplifies a frequently used teaching methodology.*
- Curriculum content and learning outcomes.
- Summary of mathematical topic (FACT BOX).
- Resources needed.

PAGES 2 & 3 **WORKSHOP ACTIVITIES FOR TEACHERS**

Two pages for you to work through with your colleagues. These activities are to be shared and discussed. For each activity there is a list of resources needed, how to organise the activity (e.g. individual, pairs, whole class), how long the activity will take, when to pause, think and try the activity, and when to record your work.

PAGES 4 & 5 **CLASSROOM ACTIVITIES FOR LEARNERS**

Two pages to help you plan your lesson. You are advised how long to allow for the activity, the resources you might need and the key questions to ask.

PAGES 6 - 10 **CHANGES IN MY CLASSROOM PRACTICE**

Pages on using the teaching strategies with additional resources and activities for use during or after the workshop such as worksheets and templates. For follow-up activities you will find lots more lesson activities on the AIMING HIGH Teacher Network
https://aiminghigh.aimssec.ac.za/category/lesson-activities/

UPPER SECONDARY **TRIGONOMETRIC IDENTITIES AND EQUATIONS**

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Vinay Kathotia has worked in mathematics education, with students, teachers and researchers, in California, Hong Kong, India, South Africa and the UK. Vinay currently lives in the UK where he has taught in secondary schools, has worked on curriculum design, education research and teacher professional development, and has led the mathematics education programmes at the Nuffield Foundation and the Royal Institution. Vinay is interested in hands-on explorations that weave together mathematics, science, art and design.

Trigonometric identities and equations

Teaching strategy: Learners presenting their ideas to an audience

Curriculum content: Derivation and use of basic trigonometric identities and reduction formulae (see Fact box below). Determining the domain (input values) for which an identity holds. Solving trigonometric equations.

Prior knowledge needed: Definitions of sin, cos, tan for angles greater than 90^{o}. To be able to represent trigonometric functions using the unit circle and graphs (including the shapes and periodic nature of the sin, cos and tan graphs). Knowing that trigonometric equations can have several solutions.

Intended Learning Outcomes At the end of this activity teachers and learners will:
- Know identities and reduction formulae for sin, cos, tan
- Understand how the identities and reduction formulae connect with the unit circle and trigonometric graphs
- Be able to simplify trigonometric expressions and solve equations using reduction formulae, identities and graphs
- Appreciate how the symmetry and periodicity of the trigonometric graphs lead to general solutions of trigonometric equations within given domains
- Have experienced deducing and communicating reduction formulae using the unit circle representation

Fact box

Identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\theta \neq 90° + 180°k,\ k \in Z$, and $\sin^2\theta + \cos^2\theta = 1$.

Reduction formulae:

$\sin(90° \pm \theta) = \cos\theta$; $\cos(90° \pm \theta) = \mp\sin\theta$;

$\sin(180° \pm \theta) = \mp\sin\theta$; $\cos(180° \pm \theta) = -\cos\theta$; $\tan(180° \pm \theta) = \pm\tan\theta$;

$\sin(270° \pm \theta) = -\cos\theta$; $\cos(270° \pm \theta) = \pm\sin\theta$; $\tan(270° \pm \theta) = \mp\cot\theta$;

$\sin(360° \pm \theta) = \pm\sin\theta$; $\cos(360° \pm \theta) = \cos\theta$; $\tan(360° \pm \theta) = \pm\tan\theta$;

$\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$; $\tan(-\theta) = -\tan\theta$.

Note that the formulae for $-\theta$ are identical to those for $360° - \theta$.

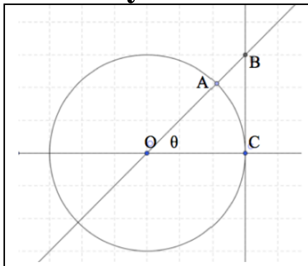
RESOURCES NEEDED: Cardboard models: circle with a pointer and two congruent ***non-isosceles*** *right-angled triangles with hypotenuse equal to the radius of the circle, preferably in different colours (one angle ≈ 25^{o} to 30^{o}). Copies of worksheet on page 10.*

Workshop Activities for teachers

Every 10 minutes, or when you finish an activity, discuss your findings with the whole group.

Activity 1 Warm-up

Pairs. A board or flip chart for sharing ideas. 5 minutes



Using the unit circle representation of sin, cos and tan (see the activity https://aiminghigh.aimssec.ac.za/grade-10-general-angle-trigonometry-investigation/ and the workshop guide 'General Angle and Periodic Functions').

How can we prove the identities below?

$\sin^2\theta + \cos^2\theta = 1$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

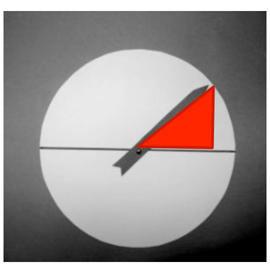
Before you read on, stop and try this for yourself.

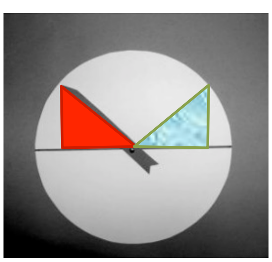
Notes to help with the activity

As $x = \cos\theta$ and $y = \sin\theta$, the identity $\sin^2\theta + \cos^2\theta = 1$ follows from Pythagoras Theorem. Draw the relevant right-angled triangle!
$\tan\theta$ can be interpreted as the slope or gradient of the radius OA, and then it follows that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. Note that both sides of the identity are not defined for $\theta = 90° + 180°k,\ k \in Z$.
Drawing examples in each of the four quadrants can reinforce that the identities are not just valid for acute angles (though the triangle we draw in each case is a right-angled triangle).

Activity 2 Seeing the Reduction Formulae in terms of rotations and reflections

Cardboard models: circle with a pointer and two congruent ***non-isosceles*** *right-angled triangles (one angle ≈ 25° to 30°), preferably in different colours, with hypotenuse equal to the radius of the circle.* *45minutes – 1 hour*





Rather than seeing the various reduction formulae as separate formulae to be memorised, they can be interpreted as arising from rotations and reflections.

Sub-activity 2.1 Carry out the following steps:
1. Start with an angle θ in the first quadrant (an acute angle) for which your triangle 'models' $x = \cos\theta$ and $y = \sin\theta$.
2. Place both triangles (one on top of the other) as shown.
3. Now rotate the pointer to $180° - \theta$.
4. Next, move the top triangle so that its perpendicular sides 'model' $\cos\theta$ and $\sin\theta$ for $180° - \theta$.

It is critical that you lift and move the original triangle as that will help reinforce whether we need a rotation or a reflection!
Describe the transformation precisely (a rotation by ….. OR a reflection in the line …..).
5. Using the two triangles, find $\cos(180° - \theta)$, $\sin(180° - \theta)$, and $\tan(180° - \theta)$.

Sub-activity 2.2: Do the above for $(180° + \theta)$, $(90° \pm \theta)$, $(270° \pm \theta)$, and $(360° \pm \theta)$, noting whether you need a reflection or a rotation in each case.

Sub-activity 2.3: What patterns do you notice in the transformations, depending on whether θ is being added to or subtracted from the fixed angle? In the case of reflections, can you relate the mirror line to the angle involved? Can you find (and prove!) any general results?

(Time permitting)

Sub-activity 2.4: Now, using the graphs of the sin, cos and tan functions, re-establish the reduction formulae.

Notes to help with the activity.

1. The use of reflections and rotations is not an additional set of rules to memorise but provides hands-on and visual experiences that can help anchor the reduction formulae for learners.
2. When working with the graphs, it can be helpful to see cosine as an 'earlier' version of sine. For example, "cos has got to 1 (90^o) earlier than sin". "What sin is now, cos was 90^o ago".
3. Review the teaching notes for the learner version of this activity as part of your discussion.

Activity 3 Solving trigonometric equations

Set of equations provided for Learner Activity 3 — 45minutes

Carry out the sorting and solving activity as described in Learner Activity 3. In pairs, one person can sort the equations into groups based on how a teacher could present the problems to the learners, and the other person can sort them based on how a 'beginning' learner may group the equations.

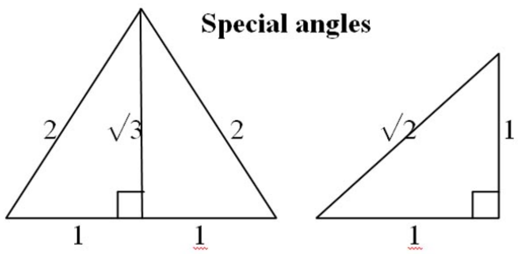
Discuss and reflect on your approaches in light of the teaching notes for this activity.

Optional activity: If you would like to work on reciprocal trigonometric functions, try out the the triangles problems in the activity 'Going round in circles' at *Underground Mathematics*: https://undergroundmathematics.org/trigonometry-triangles-to-functions/going-round-in-circles)

References: Interactive graphs for seeing the multiple solution for basic trigonometric equations are available at http://bit.ly/undergroundmathstrigequationgraphs
or https://undergroundmathematics.org/trigonometry-triangles-to-functions/general-solutions/interactive-graphs
Please note that the angles are in radians and not degrees!

Classroom Activities for Learners

Activity 1 Individual 5 minutes.



Use these two diagrams to help you write down the exact values of sin, cos and tan of the special angles 30^o, 45^o and 60^o leaving your answers in surd form.

What are the sin, cos and tan of 0^o and 90^o?

Activity 2: Finding and Sharing Reduction Formulae

Unit circle demonstration materials (see Teacher Workshop Activity 2); worksheet

Groups of 4 (2 – 4 - more); up to 6 groups present to whole class *60 – 70 minutes*

In this activity, each group of 4 learners is given an expression such as $(180° - \theta)$ and asked to find out and present the related reduction formulae for $\cos(180° - \theta)$ and $\sin(180° - \theta)$.

Half (or more) of the lesson will be spent with groups of 4 learners presenting and explaining their formulae to other learners and answering any questions.

1. Introduction of problem and examples by teacher (10 – 15 minutes)

Quick motivation: $\cos(120°) = -\sin(30°) = \frac{-1}{2}$. This can reduce calculation to acute angles!

Using the demonstration materials and steps outlined in Workshop Activity 2, show how $\cos(90° + \theta) = -\sin\theta$. Stress why the negative sign arises. Leave sin for learners but point out how it could be done. Do one further example, $\sin(180° - \theta) = \sin\theta$. Tell learners that they will be working on $\cos(180° - \theta)$ on their worksheets.

At this stage, introduce the group activity below and worksheet. If learners have not done much group work, or presented to the class, you will need to clarify and model expectations (see related points in the section Discussion of Teaching Strategy).

2. Learners work in pairs then groups of 4 on the worksheet (15 – 20 minutes)

Each group of 4 learners works on reduction formulae for one particular angle expression. Please assign one of the six choices to each group to ensure that not all will be working on the same formula. Starting with an acute angle, learners generate a conjecture for what their sin and cos formulae could be. Once both pairs have a conjecture, they discuss and agree on their formulae with another pair. They then verify these for an obtuse angle.

Next they decide how to present their formulae (who will speak, in what order, who will support the demonstration). If there is time, they can move on to other reduction formulae.

The teacher should circulate during this activity, discussing learners' approaches with them.

3. Six groups (5 minutes each) present their reduction formulae (30 – 35 minutes)

Presenters place arrows/triangles in a position that will support their explanation, then write down the start of their formula, e.g., $\cos(180° + \theta) = \ldots$. The class is asked to work out the formula based on the picture (allow 30 seconds for this). Presenters then share their reasoning, and other learners can ask questions. Have as many groups present as feasible.

Learners may recognise symmetry, reflections or rotations: encourage them to talk about these explicitly, so they make links between the geometry and the trigonometry. Depending on time and learner readiness you can formalise these ideas.

The reduction formulae for tan can be obtained using the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Activity 3: Solving Trigonometric Equations

Table of equations *Pairs then whole group* *2 × 30 minutes*

Learners sort and solve trigonometric equations, finding general solutions or solutions within a given domain. They should use known identities, graphical or unit circle representations. This can be a consolidation activity, once learners have studied such equations.

$\sin\theta = \frac{1}{2}$	$2\cos\theta + \tan\theta = \frac{1}{\cos\theta}$, $0° \le \theta \le 360°$	$2\sin\theta\cos\theta = 5\cos\theta$, $0° \le \theta \le 360°$
$\sin^2\theta = \frac{1}{2}$, $0° \le \theta \le 360°$	$\sqrt{2}\sin\theta\cos\theta + \sin\theta = 0$	$\sin 2\theta = \sqrt{3}\cos 2\theta$, $0° < \theta < 360°$
$\sin\theta + \cos\theta = 0$	$\tan 2\theta = \sqrt{3}$	$\cos^2\theta + 3\cos\theta + 2 = 0$
$\tan\theta = -1$	$\sin^2\theta + \sin\theta\cos\theta = 0$, $-360° \le \theta \le 360°$	$\cos\theta = \frac{-1}{\sqrt{2}}$, $0° \le \theta \le 720°$
$\cos\ (\theta + 90°) = \frac{-1}{2}$	$\tan(\theta - 30°) = \sqrt{3}$, $-180° \le \theta \le 360°$	$\sin 3\theta = \frac{1}{2}$, $-180° < \theta < 180°$

Task 1a (15 minutes) Learners, in pairs or small groups, are given these equations (on paper or on the board). Ask learners to group the equations into 2-4 sets based on properties they observe. If learners are not sure where to start, ask them to pick any two equations and ask, "what is the same and what is different about these equations?"

A more directed task would be to ask learners to group the equations based on how they might *solve* them. As an extension, they could construct a *flow chart* for solving such equations.

Task 1b (15 minutes) Ask one group to come to the front and show on the board how they have grouped the equations, but not explain it yet. The other learners should discuss what rules could have been used. The teacher may allow the class to ask the presenters yes/no questions to help. Ask another group to present their work so that the class can discuss and compare these.

Alternatively, one learner in each group stays at their table, to act as guide, while other learners visit the other tables and try to figure out how the equations at any table have been grouped.

Task 2a (15 minutes) Learners solve two equations, each from a different one of their sets.

Task 2b (15 minutes) Two learners are asked to present how they solved their equations, one without a restricted domain and then one with a restricted domain. They need to be as clear as possible on how they order and carry out the various steps in their solution. The class can ask questions and discuss their approaches, guided by the teacher.

Ideas for Teaching Activity 3

The grouping task helps to reveal which features of an equation learners focus on. The equations are ordered below, based on complexity. Equations1-3, 4-5, 6-7, 8-11, 12-15 form 'natural' sets.

The given equations can be solved without a calculator if learners know the trigonometric ratios for the angles 0^o, 30^o, 45^o, 60^o, 90^o. And solving the earlier equations can help with later ones. If learners have just learned to solve equations of type 1-3 below, an alternate activity would be to start with equations 6 and 9 to teach solving more complex trigonometric equations.

1. $\sin\theta = \frac{1}{2}$
2. $\tan\theta = -1$
3. $\cos\theta = \frac{-1}{\sqrt{2}}$, $0° \le \theta \le 720°$
4. $\cos\ (\theta + 90°) = \frac{-1}{2}$
5. $\tan 2\theta = \sqrt{3}$
6. $\tan(\theta - 30°) = \sqrt{3}$, $-180° \le \theta \le 360°$
7. $\sin 3\theta = \frac{1}{2}$, $-180° < \theta < 180°$
8. $\sin^2\theta = \frac{1}{2}$, $0° \le \theta \le 360°$
9. $\sqrt{2}\sin\theta\cos\theta + \sin\theta = 0$
10. $2\sin\theta\cos\theta = 5\cos\theta$, $0° \le \theta \le 360°$
11. $\cos^2\theta + 3\cos\theta + 2 = 0$
12. $\sin\theta + \cos\theta = 0$
13. $\sin^2\theta + \sin\theta\cos\theta = 0$, $-360° \le \theta \le 360°$
14. $\sin 2\theta = \sqrt{3}\cos 2\theta$, $0° < \theta < 360°$
15. $2\cos\theta + \tan\theta = \frac{1}{\cos\theta}$, $0° \le \theta \le 360°$

For equations 1-3 encourage learners to check how many solutions they expect between 0^o and 360^o and then generalise by adding suitable multiples of 360^o. Solutions can be found using interactive graphs at https://undergroundmathematics.org/trigonometry-triangles-to-functions/general-solutions/interactive-graphs but note that the website uses radians, not degrees.

Equations 4-7 can be solved by reducing them to the previous type using substitutions such as $u = 2\theta$ or $u = \theta - 30°$. Learners will need to find the related domains for u, solve for u, and then substitute back for θ. They should check that their final answer does satisfy the original equation and domain. Learners could also connect these equations with graph transformations.

Equation 8 leads to two equations ($\sin\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{-1}{\sqrt{2}}$).

Equations 9, 10 and 11 can be solved by factorising, although not all factors lead to solutions. Learners should not divide by a common factor, e.g. in example 9, $\sin\theta = 0$ gives solutions 0^o and 180^o, but dividing by $\sin\theta$ would lose these solutions as dividing by 0 is not a legitimate step.

Solving equations 12-15 requires the identities $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$.

Discussion of Teaching Strategy – Learners presenting their ideas

Learners presenting their ideas to others is a key part of both learner activities 2 and 3. It is important to learn to listen as well as to present and the activities have been designed with this in mind. If learners are new to presenting, you will need to model listening, questioning, and presenting.

To improve a learner's explanation and confidence, ask guiding questions referring to details of the explanation rather than offering an alternate explanation which could undermine the learner.

Sometimes learners may have two different but equivalent statements. Encourage learners not to assume one statement is wrong but to ask questions and listen carefully to any explanations."

In Activity 3, when learners discuss how the equations have been grouped, encourage the use of key vocabulary such as solution, equation, trigonometric function, domain, factors. Explain that this will help make it easier for them to understand each other's reasoning and ideas.

Key Questions to develop understanding - *use often with your learners*

- Describe how $\cos(90° + \theta)$ is related to $\sin\theta$. Can you do this using a diagram?
- Describe how $\cos(180° - \theta)$ is related to $\cos\theta$. Can you do this using a graph?
- For an equation … how many solutions do you expect between 0^{o} and $360°$? How do you know?
- Can you sketch the solutions using the unit circle picture?
- Can you sketch the solutions using a graph?
- If $0° < \theta < 180°$ what can you say about $\theta + 90°$?
- Describe how you could use $\sin\theta$ or $\cos\theta$ to solve $\cos(\theta + 90°) = k$.

Activities to help learners remember

Encourage learners to make quick drawings to 'recover' trig ratios of the 'special angles', rather than relying on remembering them. This supports their meaning-making.

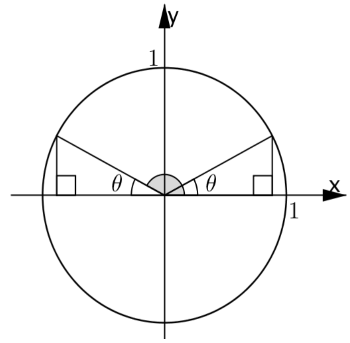
Ask learners to design a flow chart for solving a trigonometric equation and test it with a number of examples.

Ask them to 'annotate' their solutions to trig equations by labelling each line with the process they have used, e.g. 'use $\tan\theta = \frac{\sin\theta}{\cos\theta}$ ' to get an equation in just $\tan\theta$. Factorise this quadratic equation. For each solution, work out the possible values of θ. Such strategies help learner 'metacognition' and support them generalising their thinking to other similar, but different, equations. If they then explain each line to other learners, they have to voice that thinking, which helps their future thinking even more.

Use examples and applications to motivate the content. For example, reduction formulae reduce finding trigonometric values for arbitrary angles to a 'small' set of angles, 0^{o} to 45^{o}.

Trigonometric graphs and equations arise in a variety of fields – engineering, geography, medicine, music, physics, … See *Wave Trig* and other examples at
https://plus.maths.org/content/os/issue55/package/index

Worksheet for Activity 2 on Reduction Formula



A. Carry out the following steps:

1. What size is the shaded angle?

2. In the left-hand triangle, label the side that shows $\cos(180° - \theta)$.

3. Use the diagram to complete these reduction formulae:

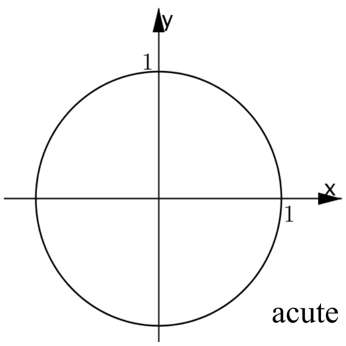
$\cos(180 - \theta) =$……………..

$\sin(180° - \theta) =$……………..

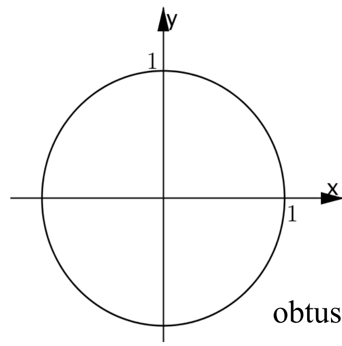
B. For one of the angles in this list

$90° - \theta$, $90° + \theta$, $180° + \theta$, $270° - \theta$, $270° + \theta$, $360° - \theta$,

draw triangles on the circles below to find reduction formulae for sin and cos of your chosen angle. You should check that your formula also works if θ is obtuse.



acute

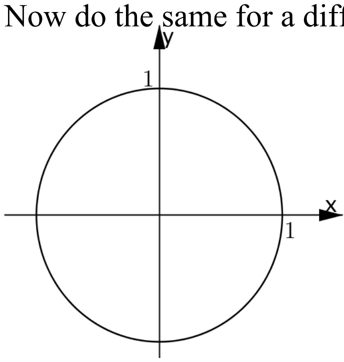


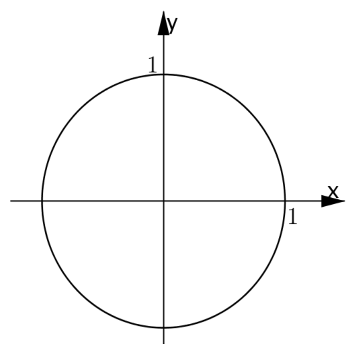
obtuse

…………………………………… ……………………………………

…………………………………… ……………………………………

Now do the same for a different angle in the list.





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