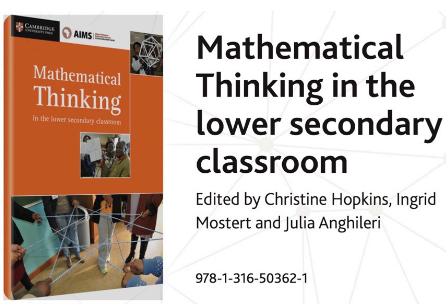
MANAGE YOUR OWN PROFESSIONAL DEVELOPMENT WORKSHOP

These guides are designed to support teachers in developing a deep understanding of the mathematics they teach and in developing more effective ways of teaching.

You can use these guides on your own or as one of a group of teachers who meet together to talk about your mathematics lessons as part of your professional development. Maybe one of you will take the lead in organizing time, date and venue but once you are doing the activities together you will all participate on equal terms in the discussion and reflection.



The Lower Secondary Workshop Guides are chapters in the AIMSSEC Mathematical Thinking Book.

Buy the book online from [Amazon] or from http://www.cambridge.org/za/education Search for AIMSSEC or for ISBN 9781316503621. To order the book in South Africa go directly to http://www.cup.co.za

For reviews and curriculum map see https://aiminghigh.aimssec.ac.za/mathematical-thinking/

Lines, circles and tangents

Developing communication and presentation skills
By Geoff Faux

Geoff Faux worked in Britain firstly as a teacher and then head of department of mathematics in comprehensive schools but after ten years moved on to be a local authority mathematics adviser. He had 42 secondary schools and 330 primary schools in his patch. At the age of 60 he joined the Open University Centre for Maths Education as an associate lecturer and spent a happy 15 years writing and tutoring maths education courses with occasional forays to Cape Town to work on AIMSEC residential courses. He is still an active member of the Association of Teachers of Mathematics (ATM). Over the years he has directed a number of national courses for them each of which has resulted in important complete issues of ‘Mathematics Teaching’. These include (129) on geometry (163) on Number, (177) on proof and (205, 206) on visualisation in mathematics.

EACH WORKSHOP GUIDE HAS A SIMILAR FORMAT:

PAGE 2 TITLE PAGE

- Teaching strategy focus. *Each guide focuses on and exemplifies a teaching methodology that is widely used.*
- Curriculum content and learning outcomes.
- Summary of mathematical topic (FACT BOX.)

PAGES 3 & 4 WORKSHOP ACTIVITIES FOR TEACHERS

Two pages for you to work through with your colleagues. These activities are to be shared and discussed. For each activity there is a list of resources needed, how to organise the activity (e.g. individual, pairs, whole class), how long the activity will take, when to pause, think and try the activity, and when to record your work.

PAGES 5 & 6 CLASSROOM ACTIVITIES FOR LEARNERS

Two pages to help you plan your lesson. You are advised how long to allow for the activity, the resources you might need and the key questions to ask.

PAGES 7 - 110 CHANGES IN MY CLASSROOM PRACTICE

Pages on using the teaching strategies with additional resources and activities for use during or after the workshop such as worksheets and templates. For follow-up activities you will find lots more lesson activities on the AIMING HIGH Teacher Network
https://aiminghigh.aimssec.ac.za/category/lesson-activities/

The AIMSSEC App can be downloaded from the internet onto any android smart phone, laptop or tablet in 3 or 4 minutes and it is free.

Go to Google Play and search for **aimssec** and follow instructions. **Please register on the AIMSSEC App so that,** when you are connected to the internet, you can add comments, ask questions and join in professional discussions on the AIMING HIGH Teacher Network website.

After downloading the AIMSSEC App, everything will be available on your own phone or other device so that you will be able to use all the AIMING HIGH resources, lesson activities and professional development workshop guides offline, that is WITHOUT USING THE INTERNET.

From time to time when you connect to the internet you can update your AIMSSEC App content with recently added resources.

Lines, circles and tangents

Developing communication and presentation skills

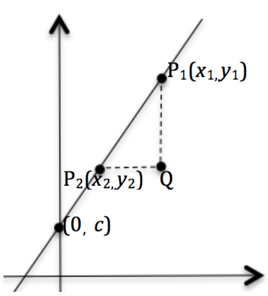
Curriculum content: Cartesian distance and gradients. Equations of straight lines, circles and tangents.

Prior knowledge needed: Know Pythagoras theorem. Know that the tangent to a circle and the radius to the point of contact are at right angles.

Intended Learning Outcomes At the end of this activity teachers and learners will:

- Know the various forms of equations of a straight line
- Understand the concepts of, and the connections between, distances, gradients, and equations of lines and circles in Cartesian coordinates
- Be able to find the equations of straight lines and circles, and of radii and tangents to circles
- Appreciate how plotting points and sketching graphs connects numerical examples with generalised definitions
- Have experienced presenting their mathematical work and ideas to their class

Fact box



Distance between two points P_1P_2 = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
This formula comes from Pythagoras Theorem for triangle P_1P_2Q.

Midpoint of line segment P_1P_2 is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Gradient of line segment P_1P_2 is $\frac{(y_1-y_2)}{(x_1-x_2)}$.

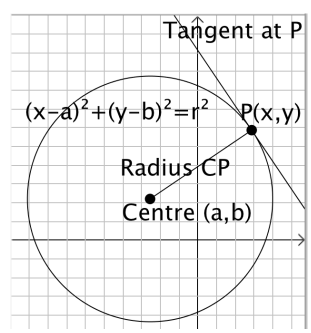
Equation of straight line between two points P_1 and P_2

The gradient of P_1P_2 is $m = \frac{(y_1-y_2)}{(x_1-x_2)}$ so the equation of the line is: $(y - y_1) = m(x - x_1)$.

This equation can be rearranged as: $y = mx + c$ where c is a constant. The line cuts the y-axis at the point $(0, c)$, called the y-intercept.

Perpendicular lines: Lines with gradients m_1 and m_2 are perpendicular if and only if $m_1m_2 = -1$

Equation of circle centre C = (a,b) radius r is $(x - a)^2 + (y - b)^2 = r^2$
To find the equations of the radius and tangent: find the gradient m of CP and its equation using this gradient and the coordinates of C or P. Then find the equation of the tangent using the gradient $-\frac{1}{m}$ and the coordinates of P.



Resources needed *Photocopies of the worksheets on pages 8 and 9.*

Workshop Activities for Teachers

Activity 1: Sorting and matching equations

Groups of four *Copy of the equations on Worksheet 1 to cut up (optional)* *30 minutes*

The challenge here is to match these equations to the edges of the rectangles marked A, B, C, D, E, F, G, H, J, K, L and M in the diagrams and to explain how you did it to the rest of the class.

1) $y = \frac{2}{3}x + 6$	2) $x + y = 4$	3) $y + 2x + 4 = 0$
4) $y = x + 4$	5) $y = \frac{1}{2}x + 6$	6) $y = -\frac{3}{2}x + 6$
7) $2y + 3x = 12$	8) $y = -2x + 6$	9) $y = -x - 4$
10) $x + y + 4 = 0$	11) $y = \frac{2}{3}x - 7$	12) $y = x - 4$
13) $y = -\frac{3}{2}x - 7$	14) $y = -x + 4$	15) $y = \frac{1}{2}x - 4$

Each group should take one of the three diagrams below and match the edges of the rectangle with at least one of the equations. *After ten minutes*: Take turns to present to the other groups the equation or equations that match each of the edges of the rectangles and give reasons.

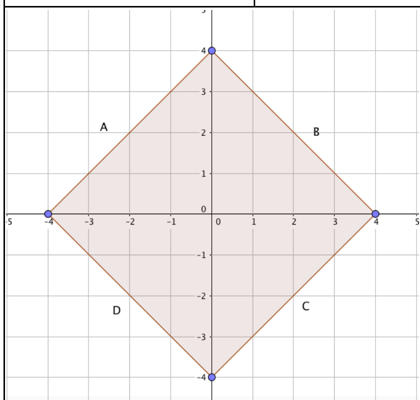


Figure 1

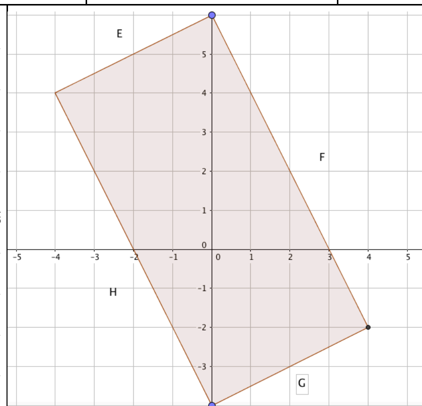


Figure 2

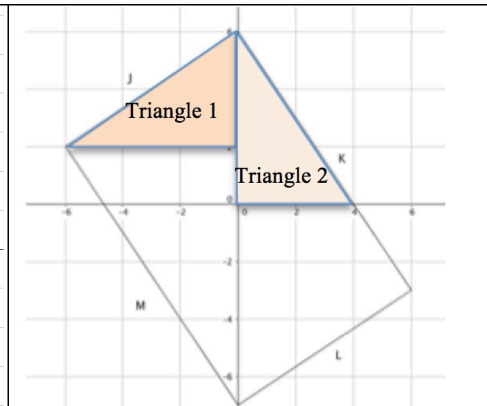


Figure 3

Notes:

(1) Pick a point on the line. Which equations are satisfied by the coordinates of the point?

(2) If talk about gradients does not occur naturally in the group discussion, the organiser can prompt by asking 'What can we say about the gradients of these lines?'

Activity 2: Distances and Gradients

Small groups *Copies of Figures 1, 2 and 3* *10 minutes*

The line J joins the points (-6 , 2) and (0 , 6). What do you notice about the distance across (in the x direction) and the distance up (in the y direction) between these two points? Explain how you can use Pythagoras Theorem to find the distance between these two points and why the distance formula given in the Fact Box on page xx is equivalent to Pythagoras Theorem.

Find the distances across and up for the two points on the perpendicular edge K at the vertices of Triangle 2? Write down the gradients of the two edges J and K. Multiply each pair of gradients together. Make a note of your answers.

For each of the rectangles in Figures 1, 2 and 3 write down the gradients of the two edges at each corner and multiply the gradients together. The group task is to work out why the answers are all the same. Will the answer always be the same when you multiply the gradients of two perpendicular lines? One of the groups should explain this to everyone else using the figures or a new figure to show why they believe it.

Activity 3: Equations of circles

Small groups *15 minutes*

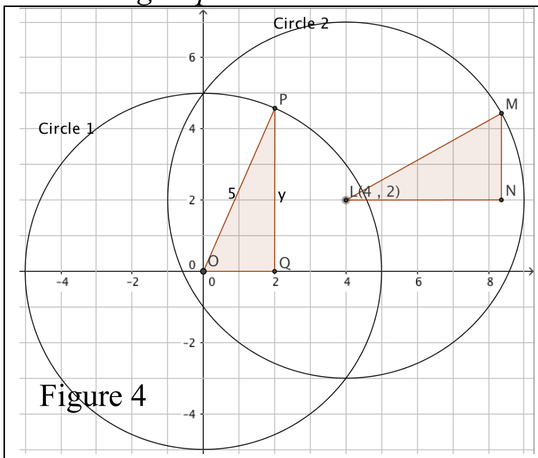


Figure 4

Circle 1 in Figure 4 has centre O(0, 0) and radius 5 units. Use Pythagoras Theorem to find the values of *y* at the point (2, y) on circle 1. Imagine that the point P moves around circle 1 and the point Q moves on the x-axis so Q is always the foot of the perpendicular from P to the x-axis. Find the y coordinates at these points on circle 1:
(1, y), (3, y), (4, y), (5, y) and (-2 , y).
Can you explain why the equation of circle 1 is $x^2 + y^2 = 5^2$?

Circle 2 has centre L(4, 2) and radius 5 units. The point M(x, y) on circle 2 can move around circle 2. Wherever M lies on circle 2 we can draw the right angled triangle LMN where LN is parallel to the x axis and MN is parallel to the y axis. Explain why the coordinates of M as it moves around circle 2 are always given by the equation $(x - 4)^2 + (y - 2)^2 = 5^2$.

Activity 4: Equations of radii and tangents

Small groups *Copy of Equation Sheet 2 to cut up (one per group – optional)* *45 minutes*

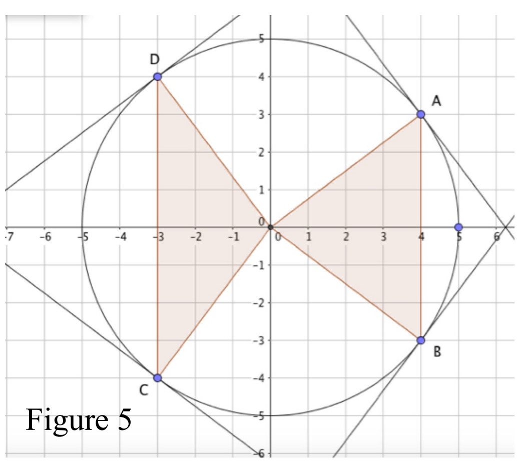


Figure 5

The points A, B, C, D in figure 6 have been chosen because they all have whole number coordinates. Match the radii and the tangents in figure 6 with the correct equations and explain why the other equations given do not correspond to these radii and tangents.

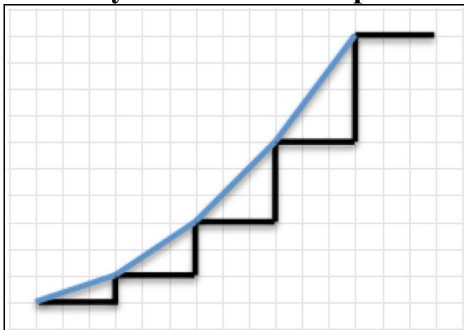
16) $y = \frac{3}{4}x$	17) $y + 3 = -\frac{4}{3}(x - 4)$
18) $y - 4 = \frac{4}{3}(x + 3)$	19) $y - 3 = -\frac{4}{3}(x - 4)$
20) $y + 4 = -\frac{3}{4}(x + 3)$	21) $y = \frac{4}{3}x$
22) $y + 3 = \frac{4}{3}(x - 4)$	23) $y = -\frac{4}{3}x$
24) $x = 0$	25) $y - 3 = -\frac{3}{4}(x - 4)$
26) $y - 4 = \frac{3}{4}(x + 3)$	27) $y = -\frac{3}{4}x$

Another group (a new group on each occasion) should explain to everyone else which equations match the radii and tangents at A, B, C, D and give their reasons. The rest should question the reasons given or suggest other possible equations that can also describe a particular line.

Classroom Activities for Learners

Activity 1: Across and Up

15 minutes



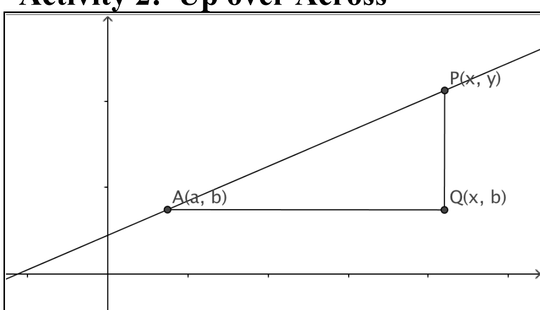
The teacher asks several learners to tell the class what they see in this image, then asks:
Who can see a curve? Who can't see a curve?
If there is no curve in the picture, what do you see?
On these stairs all the horizontals (treads) are the same length.
Are all the verticals (rises) the same length?
Can you describe how the stairs gets steeper?
Can you measure the steepness (gradient) of each step?
Can you find the lengths of the four line segments that join the top edges of the steps?
If your school has steps or stairs learners could take measurements and calculate the gradient.

Ideas for Teaching Activity 1

1. If the learners have met gradients before then this activity will check that they remember and understand the idea. If not then the learners will be familiarising themselves with the idea and relating it to their real life experience of stairs or hills.
2. The teacher can ask learners to present their work to the class and, based what the learners have done, check that they know the definition of gradient and the correct terminology.
3. In working out the segment lengths learners will use Pythagoras Theorem. This leads to understanding that the formula for the distance between two points in Cartesian coordinates is equivalent to Pythagoras Theorem.

Activity 2: Up over Across

15 minutes



A(a, b) and P(x, y) are two points on a line. Imagine that P can move along the line but that it always stays on the line.
1. Write down the gradient of AP.
2. If the gradient of the line is m, explain why the equation of the line through these two points can be written either as
$\boldsymbol{\frac{y-b}{x-a} = m}$ or as $\boldsymbol{y - b = m(x - a)}$.

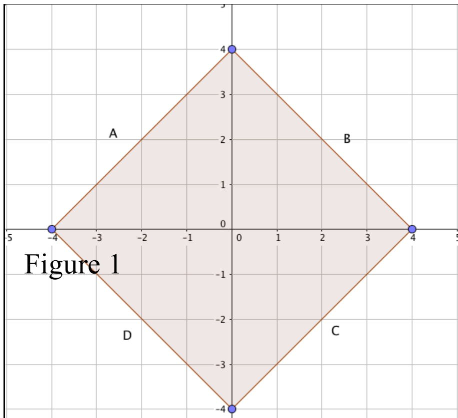
4. Explain why the equation of the line can also be written as $\boldsymbol{y = mx + c}$ where *c* is constant. What happens if $a = 0$?
5. Equations of lines can also be written in the form $\boldsymbol{px + qy + r = 0}$. What is the gradient of the line $px + qy + r = 0$ and where does it cut the *y*-axis?

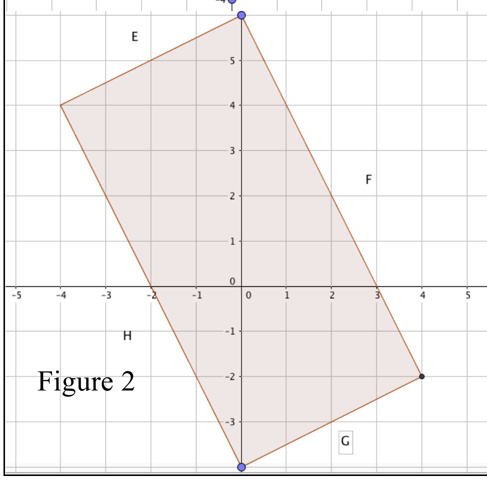
Ideas for Teaching Activity 2

Learners can work in pairs and be encouraged to talk about the reasons for each step. Then the teacher can conduct a class discussion in which learners present their ideas to the class explaining their work and finally the teacher can summarise what they have learned.
Through this activity learners will review the different forms of the equation of a straight line.

Activity 3:

Groups of 2 or 4 *Copy of Equation Sheet 1 to cut up (one per group)* *45 minutes*

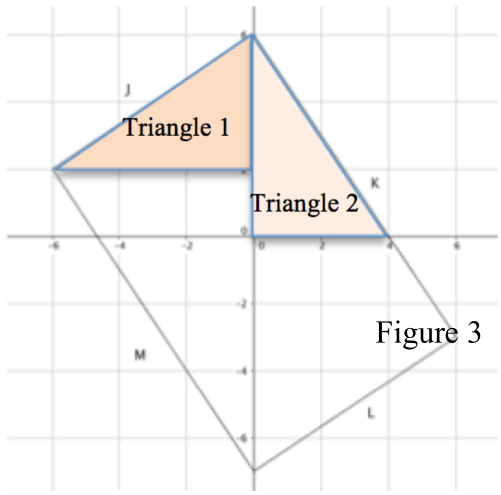




The challenge here is to find the equations on the sheet for the edges of the square marked A, B, C and D in figure 1.

Learners must then present their ideas to the class and explain why the equations they have chosen are the correct ones.

There are more equations on this sheet and when you have found the equations corresponding to the edges in figure 1 you must sort out and match the other equations to the edges of the rectangles in figures 2 and 3.



Activities 4, 5 and 6 for Learners

Activities 2, 3 and 4 for Teachers are set out in a carefully structured way. The same format could be used without any changes for activities 4, 5, and 6 for Learners. This provides material for a series of lessons that can be extended over a week or more so that time is given for developing learners communication and presentation skills.

Activity 1, 2 and 3 are in the curriculum for lower secondary but are placed here as a way of revising this work and perhaps approaching it in a different way before work on gradients of perpendicular lines (Teacher Activity 2) and on the circle and tangents (Teacher Activities 3. 4).

The link between lines drawn on graph paper and the equation of a straight line of the form $y = mx + c$ is very important generalisation. Before they can make a generalisation with any certainty learners often need to have experienced many particular examples.

Learners may have been introduced to graphs previously by making tables of values for specific equations and then plotting lines. A disadvantage of working this way is that the learners are often unaware of the generalisation they are supposed to be making or that only two points are needed to determine a straight line. They may give attention to matching the coordinates from their table of values but not give any attention to a connection with the gradient (or slope) of the line on the graph and the number m multiplying x in the general equation $y = mx + c$.

Discussion of Teaching Strategy and Lesson Planning

This unit is based upon mathematical talk and in particular **Developing communication and presentation skills** through learners presenting their ideas to an audience. Why is it important for learners to present their ideas to an audience? In presenting their ideas to others in the class learners will be forced to think about and listen to the words they use and, in listening to themselves, sharpen up their understanding of the meaning.

In planning this lesson you can sort the learners' experience into three linked parts: '**Do-Talk-Record**' (DTR)

In a classroom where DTR is in use, you are likely to see learners who are prepared:

- to think for themselves
- not to be afraid to communicate their thinking for fear it may be wrong
- to accept that wrong answers can be helpful
- to listen to their peers for comments in their own words
- to question their peers' ideas, asking for justification, examples, or proof.

(Open University, UK 2015).

When planning a lesson using the DTR framework start the lesson, or workshop, with an activity or a challenge (such as the activities given in this chapter) where the learners are actively involved. This is the **Do**ing stage

In undertaking the task you can plan for learners to meet some disturbance, challenge or inconsistency which will make them use language to sort out misunderstandings and perhaps gain comfort that others are struggling with a similar problem. 'Two heads are better than one' is a good maxim. This is the **Talk** stage

Often the best way to state a conclusion or answer is to use symbols or to write something down, before it is forgotten. This is the final **Record** stage.

As an aside; too many mathematics lessons start with the final stage, **Record**, and only track back to the earlier two when a learner becomes stuck. This may be one of the reasons why far too many learners find mathematics very difficult or impossible to use or understand.

Why do we use '**Do-Talk-Record'** in planning this workshop and lessons?

Mathematics is often about symbols and operating with symbols. Symbols are powerful because they operate without a context but, before learners can manipulate them, they need to understood the context. So the starting place is to start with a context. The context in Activity 1 and 2 is straight lines drawn on unit square graph paper. What the learners are required to **do** is match the symbol form to the straight lines. By **talk**ing to each other and perhaps hearing someone else say 'x plus y equals four is an equation I understand. I know that one and three make four, so does two plus two so (1, 3) and (2, 2) are both on that line' the listener might start to think about how this ties in with a half remembered 'y equals mx plus c' and ask where is m in $x + y = 4$? It is the talking that sorts out the equivalence between $x + y = 4$ and $y = (-1)x + 4$.

Recording does not always have to be a neat row of symbols in a book. The **record**ing in this particular activity, if you cut up the equation chart, is placing the equations beside the right lines.

Worksheet 1

1. The challenge here is to match these equations to the edges of the rectangles marked A, B, C, D, E, F, G, H, J, K, L and M in the diagrams and to explain how you did it to the rest of the class.

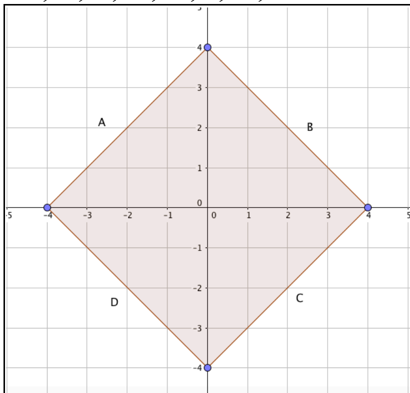


Figure 1

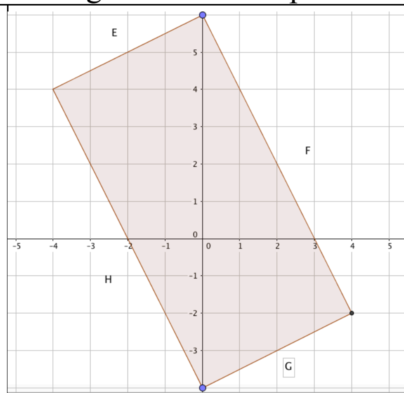


Figure 2

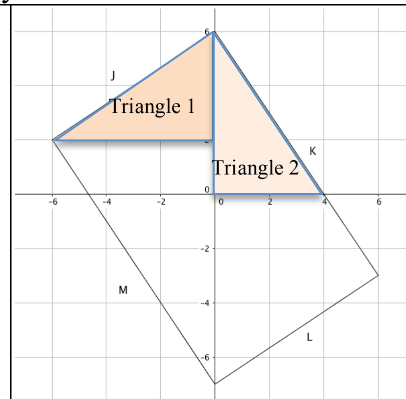


Figure 3

You can cut up the table below so that the equations can be matched to the edges of the rectangles.

1) $y = \frac{2}{3}x + 6$	**2)** $x + y = 4$	**3)** $y + 2x + 4 = 0$
4) $y = x + 4$	**5)** $y = \frac{1}{2}x + 6$	**6)** $y = -\frac{3}{2}x + 6$
7) $2y + 3x = 12$	**8)** $y = -2x + 6$	**9)** $y = -x - 4$
10) $x + y + 4 = 0$	**11)** $y = \frac{2}{3}x - 7$	**12)** $y = x - 4$
13) $y = -\frac{3}{2}x - 7$	**14)** $y = -x + 4$	**15)** $y = \frac{1}{2}x - 4$

2. The line J joins the points (-6 , 2) and (0 , 6). What do you notice about the distance across (in the x direction) and the distance up (in the y direction) between these two points? Explain how you can use Pythagoras Theorem to find the distance between these two points and why the distance formula given in the Fact Box on page xx is equivalent to Pythagoras Theorem.

3. Find the distances across and up for the two points on the perpendicular edge K at the vertices of Triangle 2? Write down the gradients of the two edges J and K. Multiply each pair of gradients together. Make a note of your answers.

4. For each of the rectangles in Figures 1, 2 and 3 write down the gradients of the two edges at each corner and multiply the gradients together. Why are all the answers the same?

5. Will the answer always be the same when you multiply the gradients of two perpendicular lines together? How does this product of gradients depend on the components of the horizontal and vertical distances between points on the lines? Explain your answer.

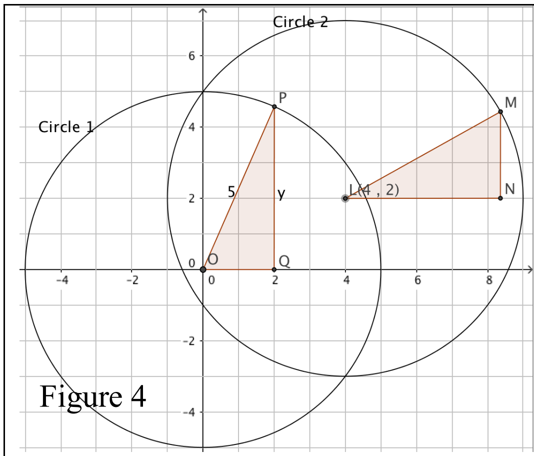


Figure 4

Worksheet 2

1. Circle 1 in Figure 4 has centre O(0, 0) and radius 5 units. Use Pythagoras Theorem to find the values of y at the point (2, y) on circle 1. Imagine that the point P moves around circle 1 and the point Q moves on the x-axis so Q is always the foot of the perpendicular from P to the x-axis. Find the y coordinates at these points on circle 1: (1, y), (3, y), (4, y), (5, y) and (-2 , y).

Explain why the equation of circle 1 is $x^2 + y^2 = 5^2$.

2. Circle 2 has centre L(4, 2) and radius 5 units. The point M(x , y) on circle 2 can move around circle 2. Wherever M lies on circle 2 we can draw the right angled triangle LMN where LN is parallel to the x axis and MN is parallel to the y axis. Explain why the coordinates of M as it moves around circle 2 are always given by the equation $(x – 4)^2 + (y – 2)^2 = 5^2$.

Worksheet 3

Match the equations to the radii to the points A, B, C and D in the diagram and to the tangents at these points.

You can cut up the table below so that the equations can be placed on the diagram corresponding to the radii and tangents.

Explain why you have chosen your equations and why the other equations do not correspond to one of the radii or tangents.

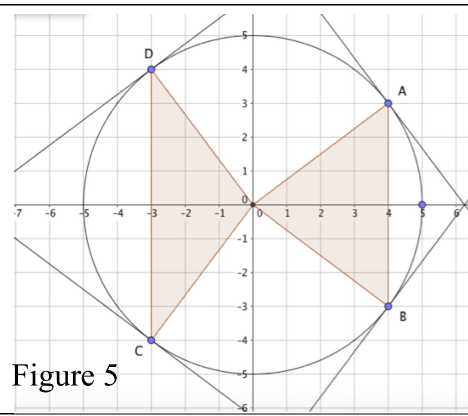


Figure 5

16) $y = \frac{3}{4}x$	17) $y + 3 = -\frac{4}{3}(x - 4)$
18) $y - 4 = \frac{4}{3}(x + 3)$	19) $y - 3 = -\frac{4}{3}(x - 4)$
20) $y + 4 = -\frac{3}{4}(x + 3)$	21) $y = \frac{4}{3}x$
22) $y + 3 = \frac{4}{3}(x - 4)$	23) $y = -\frac{4}{3}x$
24) $x = 0$	25) $y - 3 = -\frac{3}{4}(x - 4)$
26) $y - 4 = \frac{3}{4}(x + 3)$	27) $y = -\frac{3}{4}x$

ANSWERS Worksheet 1 A: 4; B: 2 & 14; C: 12; D: 9 & 10; E: 5; F:8; G:15; H:3; J:1; K:6 & 7; L:11; M:13
Worksheet 3 A: radius16, tangent 19; B: radius 27, tangent 18; C: radius 21, tangent 20; D: radius 23, tangent 26