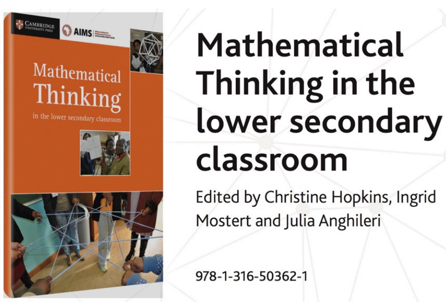
WORKHOP GUIDES FOR TEACHERS TO LEARN TOGETHER
UPPER SECONDARY AI PROBLEMS LEADING TO SERIES
Guide for your own self-help professional development workshop
and resources for inquiry based lessons.

MANAGE YOUR OWN PROFESSIONAL DEVELOPMENT WORKSHOP

These guides are designed to support teachers in developing a deep understanding of the mathematics they teach and in developing more effective ways of teaching.

You can use these guides on your own or as one of a group of teachers who meet together to talk about your mathematics lessons as part of your professional development. Maybe one of you will take the lead in organizing time, date and venue but once you are doing the activities together you will all participate on equal terms in the discussion and reflection.



These Lower Secondary Workshop Guides are chapters in the AIMSSEC Mathematical Thinking Book.

Buy the book online from [Amazon] or from http://www.cambridge.org/za/education Search for AIMSSEC or for ISBN 9781316503621. To order the book in South Africa go directly to http://www.cup.co.za

For reviews and curriculum map see https://aiminghigh.aimssec.ac.za/mathematical-thinking/

PROBLEMS LEADING TO SERIES
Representing and connecting different mathematical ideas.
By CHRISTINE HOPKINS

Christine Hopkins studied mathematics at Cambridge University and then became fascinated by the teaching of mathematics. She has worked as a teacher with children from 9 – 18 years of age and found at all ages the same spark of excitement at solving a problem or understanding something that seemed too difficult. She has worked as a teacher trainer in England at Roehampton University, and in Cambodia and Africa, trying to identify the skills and approaches which we can use and share as teachers to improve the understanding and enjoyment of mathematics by those we teach.

EACH WORKSHOP GUIDE HAS A SIMILAR FORMAT:

PAGE 1 TITLE PAGE

- Teaching strategy focus. *Each guide focuses on and exemplifies a teaching* Curriculum content and learning outcomes.
- Summary of mathematical topic (FACT BOX.)
- Resources needed

PAGES 2 & 3 WORKSHOP ACTIVITIES FOR TEACHERS

Two pages for you to work through with your colleagues. These activities are to be shared and discussed. For each activity there is a list of resources needed, how to organise the activity (e.g. individual, pairs, whole class), how long the activity will take, when to pause, think and try the activity, and when to record your work.

PAGES 4 & 5 CLASSROOM ACTIVITIES FOR LEARNERS

Two pages to help you plan your lesson. You are advised how long to allow for the activity, the resources you might need and the key questions to ask.

PAGES 6 - 10 CHANGES IN MY CLASSROOM PRACTICE

Pages on using the teaching strategies with additional resources and activities for use during or after the workshop such as worksheets and templates. For follow-up activities you will find lots more lesson activites on the AIMING HIGH Teacher Network https://aiminghigh.aimssec.ac.za/category/lesson-activities/

The AIMSSEC App can be downloaded from the internet onto any android smart phone, laptop or tablet in 3 or 4 minutes and it is free.

Go to Google Play and search for **aimssec** and follow instructions. **Please register on the AIMSSEC App so that,** when you are connected to the internet, you can add comments, ask questions and join in professional discussions on the AIMING HIGH Teacher Network website.

After downloading the AIMSSEC App, everything will be available on your own phone or other device so that you will be able to use all the AIMING HIGH resources, lesson activities and professional development workshop guides offline, that is WITHOUT USING THE INTERNET.

From time to time when you connect to the internet you can update your AIMSSEC App content with recently added resources.

Problems leading to series

Teaching Strategy: Representing and connecting different mathematical ideas.

Curriculum content: Sum of a series. Problems solving involving number patterns.

Prior knowledge: Some experience in describing patterns algebraically.

Intended Learning Outcomes At the end of this activity teachers and learners will:

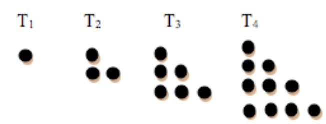
- ✓ Know how to find the sum of natural numbers
- ✓ Understand how the formula for the sum can be derived
- ✓ Be able to make conjectures and generalisations
- ✓ Appreciate the underlying structure of a pattern or problem
- ✓ Have experienced multiple representations of the same pattern.

Fact box

Notation (the way mathematics is written)

$1 + 2 + 3 + 4 + \ldots\ldots + n$ can be written more concisely as $\sum_{i=1}^{n} i$

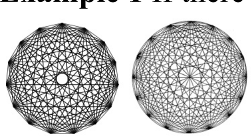
What does $\sum$ mean? This is called a sigma or summation sign and means all the terms must be added. The number underneath the sigma sign means i starts at 1, the number on top of the sigma sign means the final value of i is n which gives the *series* $1 + 2 + 3 + 4 + \ldots\ldots + n$



Conjecture: A sensible guess that tries to explain the facts, for example *"If the pattern of triangle numbers is continued then I think there will be 55 dots in the 10th triangle"*. A conjecture may be right or it may be wrong.

Generalisation: A generalisation is a general statement or formula that applies to all cases, for example
"*If the pattern of triangle numbers is continued then the nth triangle number will be* $1 + 2 + 3 + 4 + \ldots\ldots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$". Also a generalisation can be an extension of the same mathematical reasoning to higher dimensions or to other applications.

Example 1 If there are n points around a circle and every point is joined with a straight line to every



other point there will be $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$ lines. The diagram is called a Mystic Rose. If the number of points is even there will be lines through the centre of the circle and if the number of points is odd there will be an empty space around the centre of the circle.

Example 2 If a group of n people all shake hands with each other then the points on the circle model the people and the lines model the handshakes.

Resources: Mystic rose poster if available (or ask your learners to make one), compasses, protractors.

Workshop Activities for Teachers

Activity 1 Handshakes

To start the workshop everyone in the group must shake hands with everyone else and say hello even if they already know each other. Everyone must join in. When the handshaking is finished discuss how many handshakes took place. Write all the numbers suggested on the board and try to agree on the number. Repeat the handshaking and this time someone must keep a careful count.

You may like to split up into groups and discuss what the number should be. Each group should organise itself, listen to each other and probably different groups will think of different methods. Can you explain why your method works?

Everyone in the workshop should then share their ideas and compare the different methods of finding the answer. It is important to understand at least two different methods if you can.

**Can you write down an expression for the number of handshakes for 10 people, then for n people?*

Notes to help the teachers at the workshop to do the activity

1. Two people shaking hands is one handshake.
2. One way to solve the problem is for the first person to shake everyone's hand, so if there are 6 people she will shake 5 hands. Then the next person has only 4 hands to shake. The total for 6 people is $5 + 4 + 3 + 2 + 1 = 15$ handshakes and for n people it is $(n-1) + (n-2) + \ldots + 3 + 2 + 1$.
3. Another way to solve the problem is for everyone to think they must shake everyone else's hand so for six people there are 6 times 5 handshakes. But is 30 handshakes too many? If so why and how could you use this method to get the right answer?

Activity 2 Choosing Books

There are five different books in the library that you want to borrow. You can only borrow two books. How many different choices can you make? Work in pairs, choose a notation and work systematically. When you are sure of your answer compare with another pair.

**Can you see how to extend your method for 5 books to choosing 2 books from 10 books and then to choosing 2 books from n books?*

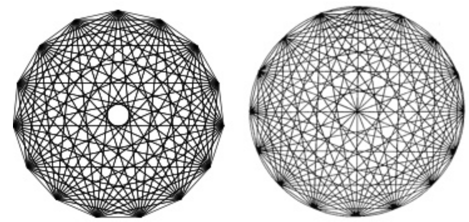
Notes to help the teachers at the workshop to do the activity

AB, AC, AD, AE	One way is to start by choosing book A and list all the pairs that include A
BC, BD, BE	then start with book B
CD, CE	then book C
DE	then book D and there are no choices left. There are $4 + 3 + 2 + 1 = 10$ ways

Another possibility is to say that the first book can be any one of 5 books and for the second book there are only 4 choices left. But this method will count both A first and then B (AB) and also B first and then A (BA) and it will count all the pairs twice so the total number of possible choices is $(5\times4)/2 = 10$

Activity 3 Mystic Rose Mystic rose poster, compasses, protractors.

Now try the Mystic rose activity from the classroom activities. It will help to have full page prints of the two diagrams on pages 7 and 8.



Key Question: Can you see why these three activities give the same answer?
Try to answer the other key questions in the text marked with asterisks *.

Discussion of teaching strategies:

- **Making connections.** This is a good example of making connections between different contexts in which the same mathematical reasoning applies. Very often, when faced with a problem, we can express the relationships involved in mathematical terms. If we then recognise that we have seen something similar before, we may already have done most of the work necessary to solve the problem, then all we need to do is to apply the same reasoning to the current context.
- **Letting the learners do the work.** By joining in this workshop you have had the chance to think carefully about the activity, to try out ideas and to understand the problem solving process. You can ruin the activity for your learners by leading them through and telling them what to do at every stage. Be patient, listen to their ideas and be confident that if you have the courage to stay quiet they will work out ideas for themselves.
- **Teaching for understanding or transformative teaching**. All mathematical formulae can be explained and understood. In the 21st century it is important to ensure that your learners understand everything that they learn. Having understood the derivation of the formula we find it easier to recall when we need to apply it. Teaching for understanding will take a little longer at first but on a practical level you will save much time at the revision stage when learners understand and are confident.
- It may seem quicker just to give learners the formula and drill them in how to apply it in the old- fashioned transmission style of teaching but later the learners are likely to become unsure and say 'I don't remember, I don't understand, I don't see why'. Because machines are much better at routine procedures than people will ever be, education that prepares learners for the world of work should equip them to understand routine procedures so that they can use machines, work more efficiently and achieve more.
- On a more idealistic level, if you teach for understanding in a transformative style, your learners will be likely to enjoy learning and to want to continue to study and enjoy mathematics.

Classroom Activities for Learners

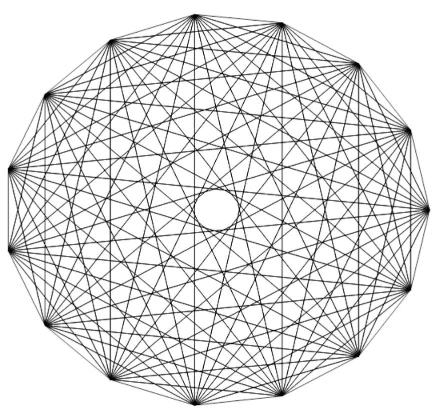
Activity 1 Handshakes

Start the lesson with the Handshakes activity. Arrange the learners in groups of 5 or 6. If they get stuck for ideas then remember that trying simple cases is a good way to tackle a difficult problem. Ask 2 learners to come to the front and shake hands: easy – one handshake. Now repeat for 3 learners, then again for 4, then 5 and let everyone record the numbers. Let the 5 learners work out for themselves how to count all the handshakes systematically and don't tell them what to do. Let other learners make suggestions. When the learners are confident that, for a group of 5, the number of handshakes is 4 + 3 + 2 + 1 or (5 x 4)/2, introduce the word **conjecture** (A sensible guess based on the evidence). Ask the learners for a conjecture as to how many handshakes there would be for 100 people. Don't say if they are right or wrong, tell them they will be able to check their conjectures later and that you will now move on to something different!

Activity 2 Analysing A Poster

Mystic rose poster 45 minutes

Mystic Rose



Put a mystic rose poster where everyone can see. Place a chair a few feet in front of the poster. Learners take turns to sit in the seat and make a short comment about the poster. Only the person in the 'hot seat' in front of the poster can speak. The comment can be general or mathematical.

**After a while ask the learners how they think the poster was drawn – do they think they could draw the poster.*

Ask the learners to draw a similar shape with just 5 or 6 points round the outside. How many lines will there be in their diagrams.

**How many lines are there in the original diagram?*

** How many lines would be in a pattern with 100 points?*

Then ask for a generalisation: **Can they give a general method for finding the answer? Can they write this as a formula?*

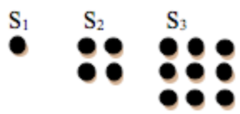
Ideas for Teaching

1. Accept all comments. Some may be about the circles people see or about liking the pattern.
2. There is often an argument about whether the poster is made up of circles or straight lines or perhaps not really circles but polygons with many sides. Eventually people will realise that the shape is entirely made up of straight lines.
3. For 6 points there are 5 + 4 + 3 + 2 + 1 = 15 lines – just like the handshake activity. Some people may work this out as 6 points to start the line and 5 points to finish the line making 6 x 5 = 30. This is too many but you are counting every line twice so you need (6 x 5)/2 lines
4. **Making connections** Mathematics allows us to make connections between problems that look very different on the surface.
   a. Number of handshakes
   b. Lines joining points on a circle
   c. Choosing two books from a shelf

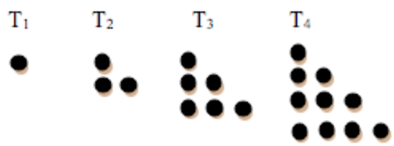
Key Question: *Can you explain why the handshake and the poster activities have the same solution? These problems have the same mathematical structure and so they have the same solution expressed as triangle numbers which are interesting because they occur in many different situations.

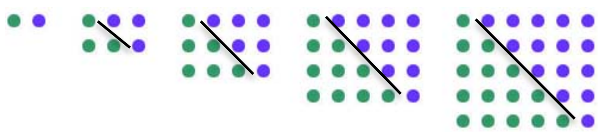
Activity 3: The sum of the first 100 natural numbers **45 minutes**

Here are diagrams representing some square numbers



and some triangle numbers





T_4	(4 x5)/2	10
T_5	(5 x6)/2	15
T_6	(6 x7)/2	21 and so on…

Questions

These diagrams represent the square numbers. How many dots will there be in S_9, in S_{100} ?
*How did you work it out?

How many dots will there be in T_9?
What about T_{100} ?

Assign different triangle numbers to different pairs of learners and ask them to answer the following questions. Collect the information on the board.
*Can you fit 2 identical triangle numbers together to form a rectangle?
*Can you use this picture to write down the formula for T_2, T_3, T_4 and T_5 ?
*Can you find a formula for T_{100}?
*Can you find a formula for T_n ?

See: https://aiminghigh.aimssec.ac.za/grades-9-to-12-triangle-number-picture/

Now the teacher tells a story

When the mathematician Carl Friedrich Gauss was a child his teacher wanted a quiet morning so he told the young children to add up all the numbers from 1 to 100. The teacher put his feet up on the desk. A few seconds later Gauss came to the front with the correct answer written on his slate.

* *How could he have worked out the answer so quickly?*

Try this! Write out this expression for T_9

$$T_9 = 1+2+3+4+5+6+7+8+9$$

and then write it out again underneath but this time write it backwards: $T_9 = 9 + 8 + 7 + \dots + 1$.
Can you see how to use Gauss's idea to find $2xT_9$ and then to find T_9 and then T_{100}?
Can you see how the patterns above illustrate Gauss's method?

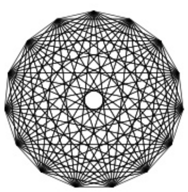
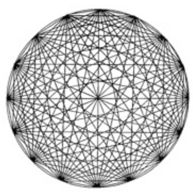
Ideas for Teaching

After the practical pattern spotting activities this is a teacher led activity. The focus is still on visualising solutions. The skill of the teacher is in asking questions and giving students time to work on the ideas for themselves. The method of writing the sum out backwards can be extended to find the sum of any arithmetic series. Confident students can use the idea of writing the terms backwards to derive for themselves the formula for the sum of a general arithmetic series as:
Number of terms x (first term + last term) divided by two.

Key Questions on Problems Leading to Summing Series.

Can you give explanations for your answers?

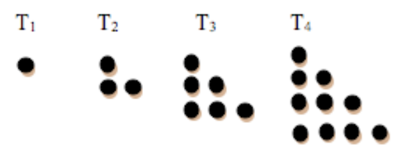
1. Can you write down an expression for the number of handshakes for 10 people if everyone in the group shakes everyone else's hand. What is the number for n people?
2. Suppose everyone thinks they must shake everyone else's hand so for six people this is 6 times 5 handshakes, but is 30 handshakes too many and if so why?
3. There are five different interesting books in the library. You can only borrow two books. How many different choices can you make? Can you extend your method to choosing two books from 10 books? What about choosing 2 books from n books.
4. How do you think that the mystic rose poster is drawn? Draw a similar shape with just 5 or 6 points round the circle. How many lines are there in your diagram?

5. How many lines are there in the diagrams on the large mystic rose posters on the next 2 pages? How do you know? Make a conjecture as to how many lines there would be in a mystic rose pattern with 100 points on the circle. Make a generalisation. Give a general formula for the number of lines in a mystic rose diagram with n points on the circumference of a circle.

6. Why do the handshakes, the book choosing and the mystic rose activities all give the same answer?

7. How many dots will there be in T_9 if this pattern of triangle numbers is continued? Can you fit 2 **identical** triangle numbers together to form a rectangle? How can you use this method to find the formula for T_{100}?

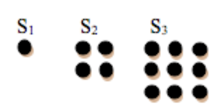


8. What is the formula for the number of dots in the nth triangle number T_n ?

9. $T_9 = 1+2+3+4+5+6+7+8+9$

   Write out this expression for T_9 and then write it out again underneath but this time write it backwards: 9 + 8 + … Can you explain how to use this method to find the formula for T_{100}?

10. These diagrams represent the square numbers. How many dots will there be in S_9? How many dots are there in S_{100} ? How did you work it out?



11. Explain why the number 1 is both a square number and a triangle number.

Further study: See also: Handshakes https://aiminghigh.aimssec.ac.za/grades-9-to-12-handshakes/ ; Triangle Number Picture https://aiminghigh.aimssec.ac.za/grades-9-to-12-triangle-number-picture/; Mystic Rose https://aiminghigh.aimssec.ac.za/grades-7-to-12-mystic-rose/ and Clever Carl www.nrich.maths.org.2478

The photo on page 2 was taken at Kingani School, near Bagamoyo in Tanzania.

Extension 1: What is special about the sorts of sequences for which Gauss's method works? What does it depend on? So could you use the same method to find 5+9+13+17+…+85? Can you make up some examples of your own where Gauss's method works?

Extension 2: Not all patterns of numbers can be summed using Gauss's method: which of the sequences in this chapter cannot? There are other patterns in number sequences that are not covered here, though, and learners will work with some when they meet compound interest. What is the pattern in this sequence 2,6,18,54,…… ? What would be the 6^{th} number in this pattern? The 10^{th}? The 100^{th}? What is the formula for the nth number in the pattern? Can learners make up some of their own sequences that work in this way? What if each number is formed by *dividing* the previous number by 3, instead of multiplying it?

Extension 3: Making graphical connections. If the numbers in a sequence are plotted against 'their 'labelling' numbers (list 1^{st} number plotted against 1, 2^{nd} number against 2, etc), on a graph, what does the pattern look like? For 'mystic rose' numbers? For sequences that can be added by Gauss's method? For the series in extension 2 above? Why?