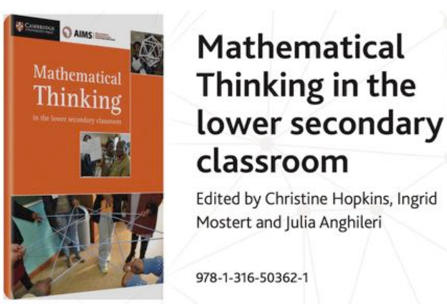
MANAGE YOUR OWN PROFESSIONAL DEVELOPMENT WORKSHOP

These guides are designed to support teachers in developing a deep understanding of the mathematics they teach and in developing more effective ways of teaching.

You can use these guides on your own or as one of a group of teachers who meet together to talk about your mathematics lessons as part of your professional development. Maybe one of you will take the lead in organizing time, date and venue but once you are doing the activities together you will all participate on equal terms in the discussion and reflection.



The Lower Secondary Workshop Guides are chapters in the AIMSSEC Mathematical Thinking Book.

Buy the book online from Amazon or from http://www.cambridge.org/za/education Search for AIMSSEC or for ISBN 9781316503621. To order the book in South Africa go directly to http://www.cup.co.za

For reviews and curriculum map see https://aiminghigh.aimssec.ac.za/mathematical-thinking/

CIRCLE THEOREMS
Learning through investigation and visualisation
By DIANE TOWNSEND

Diana Townsend taught mathematics at secondary school in the UK and to advanced European Baccalaureate level in Brussels, Belgium. She has been involved in training teachers of mathematics both in the UK and Africa, being part of the AIMSSEC Team on several occasions and also doing teacher training in Zambia.

EACH WORKSHOP GUIDE HAS A SIMILAR FORMAT:

PAGE 2 **TITLE PAGE**

- Teaching strategy focus. *Each guide focuses on and exemplifies a teaching methodology that is widely used.*
- Curriculum content and learning outcomes.
- Summary of mathematical topic (FACT BOX.)

PAGES 3 & 4 **WORKSHOP ACTIVITIES FOR TEACHERS**

Two pages for you to work through with your colleagues. These activities are to be shared and discussed. For each activity there is a list of resources needed, how to organise the activity (e.g. individual, pairs, whole class), how long the activity will take, when to pause, think and try the activity, and when to record your work.

PAGES 5 & 6 **CLASSROOM ACTIVITIES FOR LEARNERS**

Two pages to help you plan your lesson. You are advised how long to allow for the activity, the resources you might need and the key questions to ask.

PAGES 7 - 110 **CHANGES IN MY CLASSROOM PRACTICE**

Pages on using the teaching strategies with additional resources and activities for use during or after the workshop such as worksheets and templates. For follow-up activities you will find lots more lesson activities on the AIMING HIGH Teacher Network
https://aiminghigh.aimssec.ac.za/category/lesson-activities/

The AIMSSEC App can be downloaded from the internet onto any android smart phone, laptop or tablet in 3 or 4 minutes and it is free.

Go to Google Play and search for **aimssec** and follow instructions. **Please register on the AIMSSEC App so that,** when you are connected to the internet, you can add comments, ask questions and join in professional discussions on the AIMING HIGH Teacher Network website.

After downloading the AIMSSEC App, everything will be available on your own phone or other device so that you will be able to use all the AIMING HIGH resources, lesson activities and professional development workshop guides offline, that is WITHOUT USING THE INTERNET.

From time to time when you connect to the internet you can update your AIMSSEC App content with recently added resources.

Circle Angle Theorems

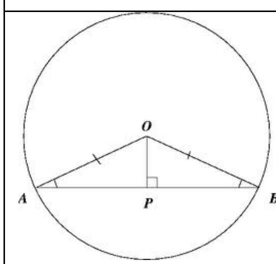
Teaching strategy: Learning through investigation and visualisation

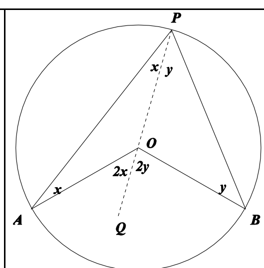
Curriculum content: Proofs of seven circle angle theorems and their converses.

Prior knowledge needed: Properties of parallel lines and similar and congruent triangles. Ability to measure angles using a protractor.

Intended Learning Outcomes At the end of this activity teachers and learners will:

- Know what is meant by an angle subtended by a chord or arc and when a quadrilateral is cyclic.
- Be able to prove that the angle subtended at the centre by an arc is twice the angle subtended at the circumference by the same arc.
- Appreciate how this and other circle-angle facts may be illustrated visually.
- Understand how to use these results to calculate subtended angles and angles of cyclic quadrilaterals.
- Experience how paper-folding can aid understanding of angles and their size.

<table>
<tr><td colspan="3">Fact Box
A tangent is perpendicular to the radius drawn at the point of contact with the circle.</td></tr>
<tr>
<td>


1. The line drawn from the centre of a circle perpendicular to a chord bisects the chord.
2. The perpendicular bisector of a chord passes through the centre of the circle.</td>
<td>


3. The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre).</td>
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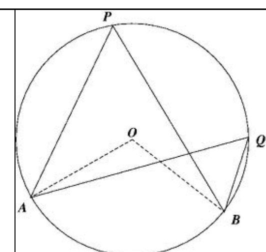

4. Angles subtended by a chord of the circle, on the same side of the chord, are equal.</td>
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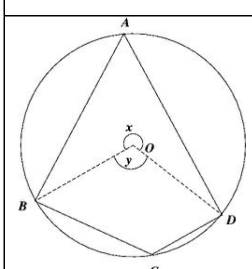
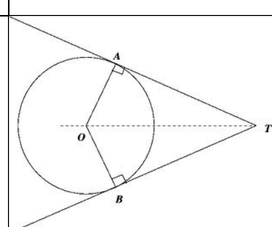
<td>


5. The opposite angles of a cyclic quadrilateral are supplementary.</td>
<td>


6. Two tangents drawn to a circle from the same point outside the circle are equal in length.</td>
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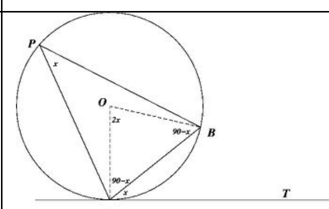


7. The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.</td>
</tr>
</table>

Workshop Activities for Teachers

The paper folding activities in this workshop guide provide experiences of discovering the properties of circles stated in the seven theorems given in the Fact Box. Folding cut-out paper circles does not prove the theorems but it suggests recipes for the formal proofs. Teachers should do all the activities before using them in lessons even if they do not have time in the workshop.

Activity 1: Finding the centre of the circle

Whole group Each person needs a cut-out paper circle (about 15cm diameter 5-10 min

(i) Fold your circle along a diameter. Mark the ends A and B.
(ii) Now unfold your circle and then fold again along a different diameter and mark the ends C and D. Mark the point of intersection of AB and CD as O. Why is this the centre?
(iii) Is there another way of finding the centre?

Notes

For (ii), does it matter which diameters are used? Repeat if necessary so that it's clear that any two diameters will intersect at the centre.
For (iii), folding the circle in half and then half again, so that A lies on B, also gives the centre and a right-angle which you will use in Activity 4.

Activity 2: Looking at the perpendicular bisector of a chord.

Whole group Cut-out paper circles with centre O marked. 5-10 minutes

(i) Fold over a segment of the circle to make a chord. Mark the ends A and B.
(ii) Fold the circle so that A and B come together. Unfold and mark the points where this line cuts the circle as C and D.
(iii) If AB and CD intersect at P, what do you know about AP and PB? What about angle CPA?
(iv) How is CD related to AB?
(v) Is CD a diameter of the circle?

Notes

In (iii) we're looking for the recognition that CD is the perpendicular bisector of the chord AB.
In (v) the fold-line CD should pass through the centre O. This demonstrates the theorem that the perpendicular bisector of a chord passes through the centre.

To prove Theorem 1 use the RHS congruency condition for Δs OAP & OBP and deduce AP=PB.
To prove Theorem 2 use the SSS congruency condition for Δs OAP & OBP and deduce OP$\perp$AB.

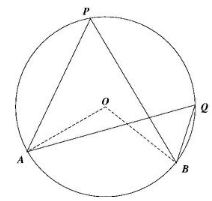
Activity 3: **Angles at the centre and circumference subtended by the same arc/chord.**

Work in pairs Cut-out paper circles with centre O marked. 10-12 minutes

(i) You and your partner should have identical circles and mark identical points. Both mark two points A and B one side of O and a point P the other side of O.
(ii) One of you folds along the chords AP and BP, the other AO and BO.
(iii) By folding again so that AO and BO are aligned, make the angle ½ AOB.
(iv) Compare this angle ½ AOB with <APB.

Notes

This illustrates the angle subtended at the centre being twice the angle subtended at the circumference. In (iii) it may be easier to cut out <AOB.



Since P is any point on the circumference it can be deduced that all angles subtended at the circumference by the same arc are equal. (Theorem 4) The formal proof of Theorem 3 is given on page 8 and Theorem 4 follows immediately from Theorem 3.

Activity 4: The angle on the diameter.

Work in pairs | Cut-out paper circle with centre O marked | 5 -10 minutes

(i) Fold your circle to make a diameter and mark the ends A and B
(ii) Mark a point P on the circumference.
(iii) Fold along AP and BP to show < APB. Compare with your neighbour.
(iv) Compare with the right-angle you have from the Activity 1.
(v) Can you see a link with the result from Activity 3?

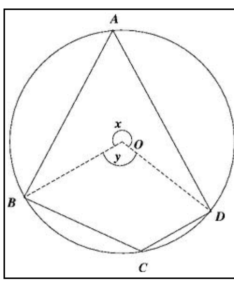
Notes
Here the angle subtended at the centre is on the diameter itself so it is 180°. Since half of this is 90°, we should expect the angle at the circumference to be 90°

Activity 5: The opposite angles of a cyclic quadrilateral are supplementary.

Work in pairs | Cut-out paper circle with centre O marked | 20 minutes

(i) Both partners need to mark 4 points on the circumference so they have identical cyclic quadrilaterals, ABCD.
(ii) Fold along the chords AB, BC, CD, DA.
(iii) Using one angle from each quadrilateral, place <A adjacent to <C. What do you notice?
(iv) Using one angle from each quadrilateral, place <B adjacent to <D. What do you notice?

Notes
The angles A and C should line up to make a straight line, i.e. <A + <C = 180°, and similarly for <B and <D. This activity demonstrates Theorem 5.
You could extend this to note that the exterior angle of a cyclic quadrilateral is equal to the interior opposite.



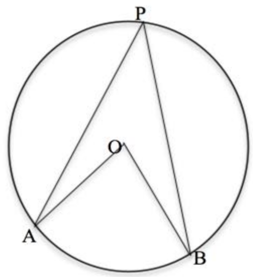
Theorem 5 If ABCD is a cyclic quadrilateral then $\angle BAD + \angle BCD = 180^o$.
Proof Draw radii OB and OD.
By Theorem 3
$\angle BAD = ½ y$
$\angle BCD = ½ x$
$x + y = 360^o$ (angles around a point).
Hence $\angle BAD + \angle BCD = 180^o$, that is the opposite angles are supplementary.

Classroom Activities for Learners

Activity 1: Angles subtended by the same arc at the circumference and centre of a circle.

Individually then pairs, then whole class discussion.

Compasses, rulers, pencils and protractors. 10 minutes.



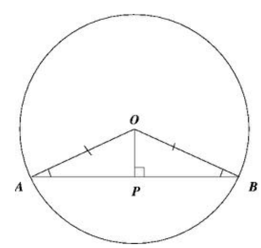
The teacher draws this diagram on the board and tells the learners to copy it carefully, marking the centre O, and A, P and B on the circumference of the circle but that it does not matter exactly where. Then all the learners should measure angles APB and AOB and write down the measures and anything that they notice. Then they should compare their results with their partners.

Notes

Although the learners may not measure the angles accurately most of them should notice that ∠AOB is double ∠APB. As different learners will have drawn A, P and B in different positions it should be apparent from class discussion that it looks as if ∠AOB is always double ∠APB (Theorem 3) but this is only a conjecture and still needs to be proved. To reinforce this conjecture the class should do the paper folding Activities 1, 2, 3 and 4 from the teacher workshop. A class could do this in 45-60 minutes and all the activities in about 2 hours.

Activity 2: Theorems and Converses.

30 minutes.

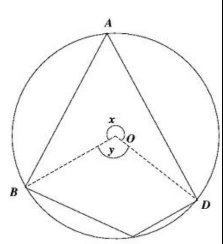
Teacher	**Learners**
The teacher asks the learners if the following pairs of statements mean the same and to say whether the statements are true or false. What do they notice about the pairs of statements? Statement A1: If it is a dog then it is not a cat. Statement A2: If it is not a cat then it is a dog. Statement B1: If it rains then I take my umbrella. Statement C2: If I take my umbrella then it rains. Statement C1: If $x = 5$ then $x > 1$. Statement C2: If $x > 1$ then $x = 5$. Statement D1: If $x = -2$ then $x^2 = 4$. Statement D1: If $x^2 = 4$ then $x = -2$. The teacher explains that each statement is of the form "If p then q" where p and q are component statements and that, in all the pairs, each statement is the **converse** of the other. 2. The teacher explains that the triangles are congruent (RHS) so it follows that AP=PB.	1. Pairs of learners discuss the 4 pairs of statements, whether each statement is true or false and whether they mean the same thing. 2. Some learners may notice that, in each pair, the second statement reverses the implication, so "If p then q" becomes "If q then p". Then the whole class discusses the statements and agrees that the pairs of statements do not mean the same thing. 2. In the diagram AB is a chord of the circle and OA and OB are radii. Pairs of learners discuss triangles ΔOAP and ΔOBP and try to prove the following theorem.  **Theorem 1** If OP⊥AB then AP=PB.

3. The teacher invites suggestions from the class, summarises their ideas and makes it clear that: • Sometimes both theorem and converse are true (as in Theorem 1) but this is not always the case so both have to be proved. • The Converse of Theorem 1 is If AP=PB then OP⊥AB and that • the proof is: ΔOAP is congruent to ΔOBP (SSS) so ∠OPA=∠OPB=90°.	3. Learners work in pairs to state and prove the converse of Theorem 1.

Notes
Learners should have the experience of thinking about the ideas for themselves and discussing them so that the teacher can help them to develop their thinking and can use the learners' ideas in giving a final explanation and summary of what they have learned in the lesson.

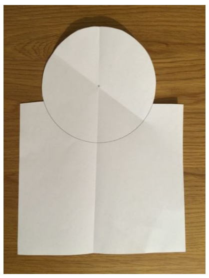
Activity 3: The opposite angles of a cyclic quadrilateral.

Cut-out paper circles with centre O marked Work in pairs. 20 minutes

Teacher	**Learners**
Teachers may write the instructions on the board or on a worksheet. They should go around helping the learners as necessary and asking questions to encourage the learners to think about the opposite angles of the quadrilateral. Based on the learners observations in class discussion, teachers should make sure that all the learners understand that angles A and C should line up to make a straight line, that is <A + <C = 180°, and similarly for <B and <D. Draw this diagram on the board and lead the class by question and answer to prove *Theorem 5* If ABCD is a cyclic quadrilateral then ∠BAD + ∠BCD = 180°. *Proof* Draw radii OB & OD. By Theorem 3 ∠BAD = ½ y ∠BCD = ½ x x + y = 360° so ∠BAD + ∠BCD = 180°.  Extend this to show that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. Ask the learners to discuss and write down the statement of the converse of Theorem 5.	(i) Both partners should mark 4 points on the circumference so they have identical cyclic quadrilaterals, ABCD. (ii) Fold along the chords AB, BC, CD, DA. (iii) Using one angle from each quadrilateral, place <A adjacent to <C. What do you notice? (iv) Using one angle from each quadrilateral, place <B adjacent to <D. What do you notice? Learners copy the statement and proof of Theorem 5 into their notebooks. Pairs of learners discuss and write down the statement of the converse of Theorem 5. For homework learners do the worksheet on page 7.

Discussion of Teaching Strategy

Seven theorems are discussed in this chapter because they all occur in the curricula in many countries. By folding cut out paper circles learners can visualize the positions and relationships between the lines and angles in the circle theorems. These activities illustrate the properties stated in the theorems and the activities lead to visualization that helps learners to gain a deeper understanding of the geometrical properties and helps them to remember these properties and to apply them to solving geometrical problems. The sequences of steps in the paper folding activities can be described and written as formal statements with reasons that together build formal proofs of the theorems.



Similar paper folding activities can easily be devised that illustrate Theorem 6: the tangents drawn to a circle from a point outside the circle are equal in length and Theorem 7: the angle between a chord and the tangent to the circle at the endpoint of the cord is equal to the angle subtended by that chord in the alternate segment. Instead of cut-out paper circles these activities use a part cut-out paper circle as shown in the photograph.

The teaching strategy should be to guide the learners through these practical investigations, encouraging them to notice certain properties and to make conjectures about what might be true in general. Another way to gather evidence to support the conjectures is for the learners to draw diagrams and to measure lengths and angles (as in Classroom Activity 1). Teachers may give the drawing and measurement reinforcement activities for the other theorems as extra support to learners who struggle with this topic. It is important to avoid the misconception that observation of even a large number of cases supporting the conjecture proves it is true. The supporting evidence does not prove that the conjecture is true in all cases and a formal proof is needed.

Another way to reinforce the benefits of visualization, and to help learners to remember these concepts, is for the class to make a large poster of the seven theorems for the classroom wall. The theorems and their converses should be stated on the poster with diagrams, and the cut-out paper circles can be pasted onto the poster. The teacher should refer to this from time to time as learners engage in further work on geometry.

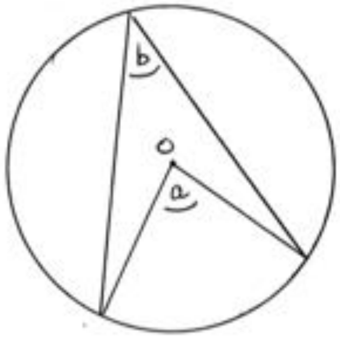
Key Questions to develop understanding

Do you notice any equal angles and why are they equal?
Can you see any right angles?
Do any angles add up to 90^o or to 180^o? Why?
Are there any congruent triangles in the diagram? How do you know?
Are there any similar triangles in the diagram? How do you know?
Can you see any equal lengths? How do you know they are equal? What does this tell you?

To check your knowledge and understanding – *try these questions*

In the following diagrams, O is the centre of the circle.
Calculate the angles as indicated:

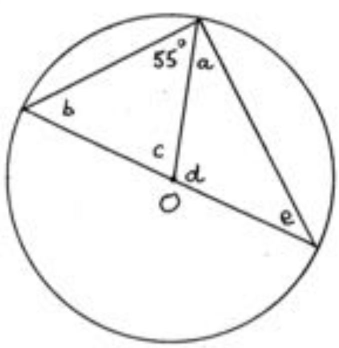
1. Find b if (i) a = 86° (ii) a = 76°
   Find a if (iii) b = 35° (iv) b = 27°



(i) b = (ii) b =

(ii) a= (iv) a =

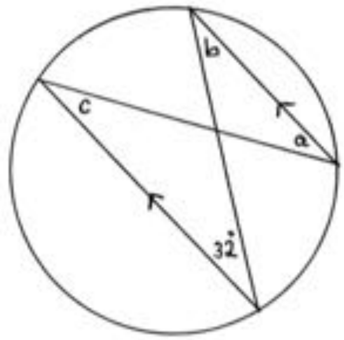
2.



a = , b = , c =

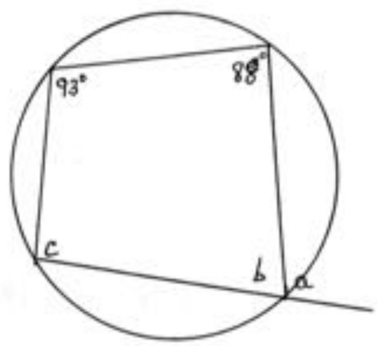
d = , e =

3.



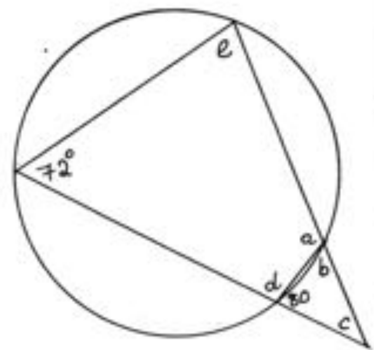
a = , b = , c =

4.



a = , b = , c =

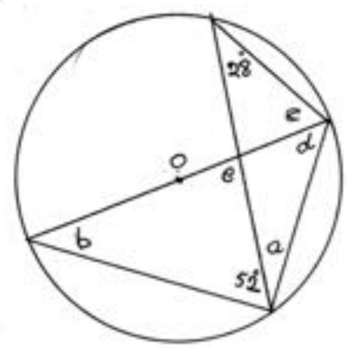
5.



a = , b = , c =

d = , e =

6.

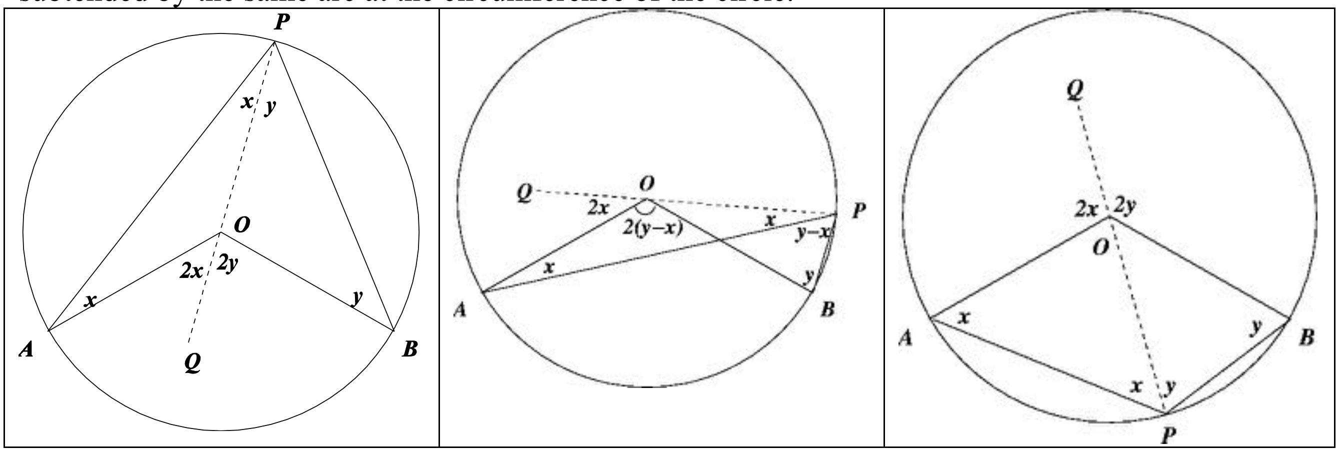


a = , b = , c =

d = , e =

Proofs of Theorems

Theorem 3 The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circumference of the circle.



Proof Three diagrams are necessary here to show the possible configurations but the proof is the same in each case.

As the radii are equal and the base angles of an isosceles triangle are equal:

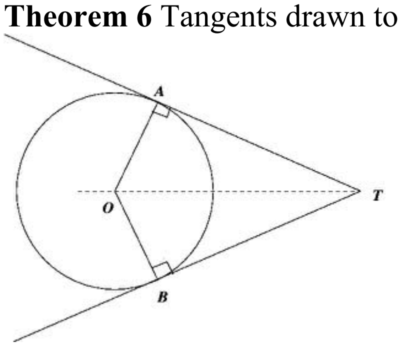
∠OAP=∠OPA=x and ∠OBP=∠OPB=y.

As the exterior angle of a triangle equals the sum of the two interior opposite angles:

∠AOQ=2x and ∠BOQ = 2y.

Hence ∠AOB = 2∠APB.

Theorem 6 Tangents drawn to a circle from the same point outside the circle are equal in length.



Proof

In ΔOAT and ΔOBT

OA = OB

∠OAT = ∠OBT = 90^o

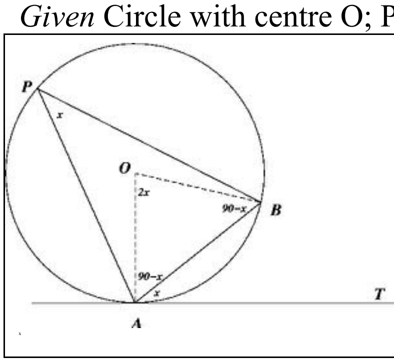
OT is common to both triangles

ΔOAT ≡ ΔOBT *(RHS)*

Hence TA = TB

Theorem 7 The angle between a tangent, and a chord drawn to the point of contact of the chord, is equal to the angle that the chord subtends in the alternate segment.

Given Circle with centre O; PT a tangent at P; PB a chord; A is a point on the circumference.

	*Proof* Let ∠BAT = x , then ∠OAB = 90 – x because the radius is perpendicular to the tangent. Triangle OAB is isosceles and so ∠OBA = 90 – x and ∠AOB = 2x. By Theorem 3, ∠APB = x. Hence ∠BAT= ∠APB = x.	**Converse** If ∠BAT= ∠APB then AT is tangent to the circle at A. *Proof* If ∠APB = x then ∠AOB = 2x. Triangle OAB is isosceles and so ∠OAB = ∠OBA = 90 – x As ∠BAT= x so ∠OAT=90^o and OA ⊥ AT. So AT is a tangent to the circle at A.