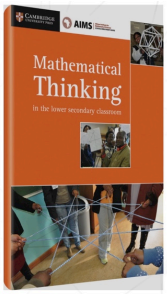
MANAGE YOUR OWN PROFESSIONAL DEVELOPMENT WORKSHOP

These guides are designed to support teachers in developing a deep understanding of the mathematics they teach and in developing more effective ways of teaching.

You can use these guides on your own or as one of a group of teachers who meet together to talk about your mathematics lessons as part of your professional development. Maybe one of you will take the lead in organizing time, date and venue but once you are doing the activities together you will all participate on equal terms in the discussion and reflection.



Mathematical Thinking in the lower secondary classroom

Edited by Christine Hopkins, Ingrid Mostert and Julia Anghileri

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The Lower Secondary Workshop Guides are chapters in the AIMSSEC Mathematical Thinking Book.

Buy the book online from Amazon or from http://www.cambridge.org/za/education Search for AIMSSEC or for ISBN 9781316503621. To order the book in South Africa go directly to http://www.cup.co.za

For reviews and curriculum map see https://aiminghigh.aimssec.ac.za/mathematical-thinking/

EXPONENTS AND SURDS

Developing Mathematical Language and Understanding

By Corinne Angier

Corinne has taught on many AIMSSEC courses and, for twenty years, Corinne alternated between teaching mathematics in secondary schools in England and working as a teacher trainer. She has worked on a national development project for primary teachers and with the charity Teach First. She now works as a mathematics consultant for a trust which trains teachers and hosts a Maths Hub that promotes teacher development. It is the home of White Rose maths:
https://www.tes.com/teaching-resources/teaching-for-mastery-in-primary-maths/whiterosemaths

Corinne now teaches in a vocational college where her students are preparing to re-sit the mathematics papers in the General Certificate of Secondary Education (GCSE) because they have done badly in their previous attempts. The GCSE is the public examination taken by secondary-school students of about age 16 in England, Wales, and Northern Ireland.

EACH WORKSHOP GUIDE HAS A SIMILAR FORMAT:

PAGE 2 **TITLE PAGE**

- Teaching strategy focus. *Each guide focuses on and exemplifies a teaching methodology that is widely used.*
- Curriculum content and learning outcomes.
- Summary of mathematical topic (FACT BOX.)

PAGES 3 & 4 **WORKSHOP ACTIVITIES FOR TEACHERS**

Two pages for you to work through with your colleagues. These activities are to be shared and discussed. For each activity there is a list of resources needed, how to organise the activity (e.g. individual, pairs, whole class), how long the activity will take, when to pause, think and try the activity, and when to record your work.

PAGES 5 & 6 **CLASSROOM ACTIVITIES FOR LEARNERS**

Two pages to help you plan your lesson. You are advised how long to allow for the activity, the resources you might need and the key questions to ask.

PAGES 7 - 110 **CHANGES IN MY CLASSROOM PRACTICE**

Pages on using the teaching strategies with additional resources and activities for use during or after the workshop such as worksheets and templates. For follow-up activities you will find lots more lesson activities on the AIMING HIGH Teacher Network
https://aiminghigh.aimssec.ac.za/category/lesson-activities/

The AIMSSEC App can be downloaded from the internet onto any android smart phone, laptop or tablet in 3 or 4 minutes and it is free.

Go to Google Play and search for **aimssec** and follow instructions. **Please register on the AIMSSEC App so that,** when you are connected to the internet, you can add comments, ask questions and join in professional discussions on the AIMING HIGH Teacher Network website.

After downloading the AIMSSEC App, everything will be available on your own phone or other device so that you will be able to use all the AIMING HIGH resources, lesson activities and professional development workshop guides offline, that is WITHOUT USING THE INTERNET.

From time to time when you connect to the internet you can update your AIMSSEC App content with recently added resources.

Exponents and Surds

Teaching Strategy Focus: Developing mathematical language and understanding.

Curriculum content: Use of the laws of integer exponents and of surds to simplify expressions and to solve equations.

Prior knowledge needed: Knowledge of exponent notation for small positive integer exponents, square roots and cube roots.

Intended Learning Outcomes At the end of this activity teachers and learners will:

- Know the conventions for writing powers and that negative powers are reciprocals and know the fractional powers for square roots and cube roots.
- Understand the rules for operating with exponents and surds
- Be able to solve problems involving exponents and surds and explain their solutions using precise vocabulary
- Appreciate that it is easier to carry out a standard procedure than to explain to someone else why a method works
- Have experienced explaining their solutions to other learners and to the whole class.

Fact box

Definitions

For any real number a multiplied by itself n times we write $a \times a \times a \times a \ldots \times a = a^n$, where the integer n is called the **exponent**, and alternatively the **power** or the **index**.

$a^0 = 1$ for $a \neq 0$

$a^{-n} = \frac{1}{a^n}$ for $a \neq 0$

$a^{\frac{1}{2}} = \sqrt[2]{a}$ and $a^{\frac{1}{3}} = \sqrt[3]{a}$

Note: In general $a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ $a > 0$, $n > 0$ but here we use only m, n = 2 or 3.

From the above definitions it follows that, for real numbers a and b and integers m and n :

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$

$a^m \times b^m = (ab)^m$

List of resources needed for activities in the chapter:

- Copies of card set 1 for each group of participants (optional).
- Copies of card set 2 for the Power Matching Classroom Activity 1 for each group.
- Scissors.

Workshop Activities for Teachers

Activity 1: Working through and discussing lesson activities.

Pairs followed by whole group discussion *40 minutes*

Work through and discuss Classroom Activity 2 - Explore 64

Notes and discussion points:

- **Formative assessment**: The two activities in the Classroom Activities for Learners section provide a means of revising work that learners have done in earlier years. The formal rules are not needed at this stage. While the learners are working on Classroom Activities 1 and 2 the teacher can observe what they can and cannot do and what the learners remember and understand. Learners can work in pairs or threes and help each other.
- If some learners find the task too challenging, then it is useful to have some alternative tasks that will help them to build up their understanding. For example:
  - *"Work out 2^3. So what does this tell you about the cube root of 8? How can it be written?"*
  - *"Work out 3^2. So what does this tell you about the square root of 9? How can it be written?"*
  - *"Work out 4^3. So what does this tell you about the cube root of 64? How can it be written?"*
- To explain why $2^0 = 1$ and $2^{-1} = ½$ it may help to write $2^6 = 64$ and divide repeatedly by 2 as follows and keep asking questions so that the learners tell you the powers of 2 that give 64, 32, 16 etc.:

	÷2	÷2	÷2	÷2	÷2	÷2	÷2	÷2
64	32	16	8	4	2	1	½	¼
2^6	2^5	2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}

- Practice some ways you could help learners by asking them questions and giving hints, e.g. *'Is there another way we could write* $64^{\frac{1}{2}}$ *?'*
- If fractional indices are not understood then learners could be asked to find values of $(64^{\frac{1}{2}})^2$, $(64^{\frac{1}{3}})^2$, $(64^2)^{\frac{1}{3}}$ and to write all these as powers of 2.
- Write down some difficult questions that learners might ask, e.g. *'What does* $64^{\frac{2}{3}}$ *mean?'*
- Share with other teachers ways to answer these learners' questions.
- All the activities in this Teacher Workshop session are also suitable for learners.

Activity 2: Building from the known

Groups of 2 or 3 *40 minutes*

Refer to the definitions in the fact box (page 1) which tell you how to write down what you do with numbers when you repeatedly multiply the numbers by themselves, divide them or take roots.

Take each of the statements on the right in turn and discuss in pairs how, from the definitions, it follows that the left hand side is equal to the right hand side.

$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$(a^m)^n = a^{mn}$$
$$a^m \times b^m = (ab)^m$$

Then simplify the following expressions and find the odd one out in each line. Can you explain why it's the odd one out?

a. $3x^3 \times 2x^4 \times 5y^2$ $10y^2x^7$ $15yx^2 \times 2yx^5$

b. $\frac{12x^{10}}{3x^2}$ $\frac{2x^3 \times 6x^7}{3x^2}$ $2x^2 \times 6x^4$

c. $(3x^{\frac{1}{2}})^3$ $(27x^3)^{\frac{1}{2}}$ $27(x^3)^{\frac{1}{2}}$

Notes:

This is a good activity for formative assessment that you can use with older learners to check their knowledge and understanding.

Activity 3: Helping a confused learner - Mzu's disastrous homework

Pairs *30 minutes*

Sadly Mzu has got every single one of his homework questions wrong!

a. Substitute his answers back into the equations to show that they do not work.

b. Work with your partner. Take it in turns to be the teacher and to be Mzu. The person who is the teacher should try to help Mzu to understand why his solutions are wrong. The person who is Mzu should try to explain his or her reasoning.

c. Write the correct solutions to the homework questions and next to each one write some comments and reminders to explain the correct strategy.

Question 1	Question 2
$\sqrt{x-9} = 4$	$3^x = \frac{1}{81}$
Solution: $\sqrt{x} - \sqrt{9} = 4$ $\sqrt{x} = 4 + \sqrt{9}$ $\sqrt{x} = 4 + 3$ $\sqrt{x} = 7$ $x = 49$	Solution: $3^x = \frac{1}{3^3}$ $3^x = 3^{-3}$ $x = -3$

Notes and discussion points:

- In 'Mzu's homework activity' part b is the most important as this is an opportunity to talk about misconceptions and to explain the correct reasoning. When learners do this activity the written work in part c will reflect how well they have managed to develop a conceptual understanding of exponents.
- Working through the learners' tasks with other teachers often reveals different ways of thinking about the mathematics and helps to prepare for teaching a topic. Teachers who have taught the topic before may describe issues that have arisen and the group can discuss ways of responding.
- Make a note of everything that you think learners will struggle with.
- Practice some ways you could help learners by asking them questions and giving hints.

Classroom Activities for Learners

Activity 1: Power Matching Puzzle

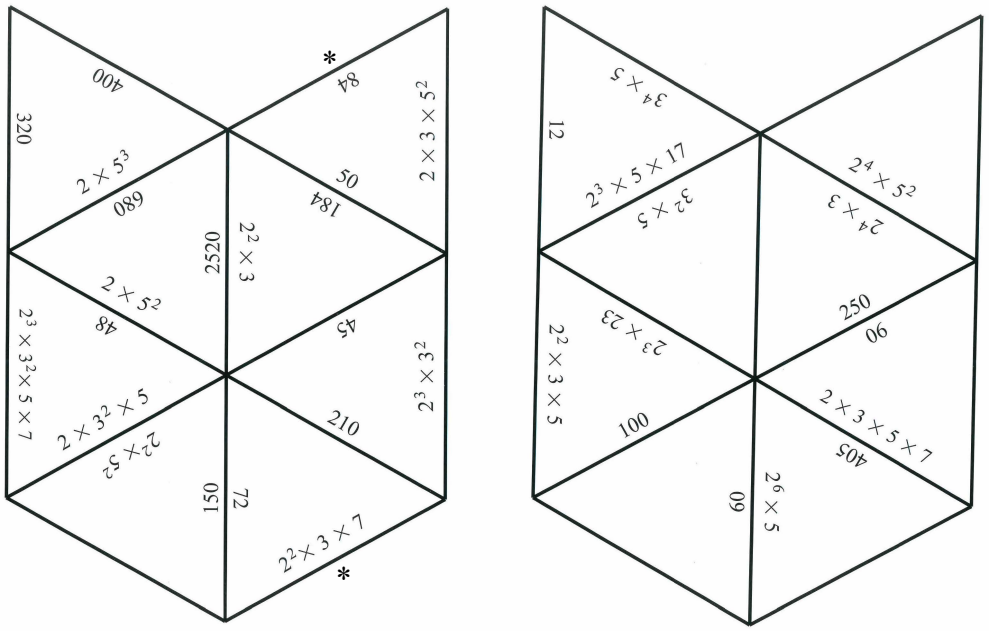
Copy of Card set 2 for each group, scissors *Pairs or small groups* *30 minutes*

You will need the set of 16 small triangles shown below (and if necessary to cut them out).

You must make one large triangle by placing the 16 small triangles so that the numbers on the matching edges are the same, for example: $2^2 \times 3 \times 7 = 84$ so these two edges (marked *) must be placed together.

Work out all these expressions involving exponents.

Find the matching numbers and place them edge to edge.



Notes

If necessary the teacher can ask the following questions to guide the learners in doing this activity:

- What do you notice about the pieces that only have numbers on **one edge**? Where do you think they go in the final triangular arrangement?
- What about the pieces with numbers on **2 edges**? Where do they go in the final triangle?

Activity 2: Explore 64

Pairs then whole class *Showboards, Card set 1 (optional)* *50 minutes*

STEP 1. What do you know about the number 64?

Write down as many facts as you can think of about the number 64.

For example:

$64 = 8 \times 8$,

$64 = 8^2$ and we say "64 equals 8 squared," or "64 equals 8 to the power 2", or "64 equals 8 to the exponent 2".

Remember that roots are written as fractional indices, for example $\sqrt{64} = 64^{\frac{1}{2}} = 8$

Work with a partner. Then the whole class should share their ideas and make a list on the board of all the facts they know about 64 that can be written and talked about in different ways.

STEP 2

1. Calculate the value of the expressions below.

2. Can you explain why some of the expressions give you the same answer?

a. $64^{\frac{1}{2}}$	**b.** $64^{\frac{1}{3}}$	**c.** $64^{\frac{2}{3}}$	**d.** $64^{\frac{3}{2}}$	**e.** 64^{0}
f. 64^{1}	**g.** $(64^{\frac{1}{2}})^{3}$	**h.** $(64^{\frac{1}{3}})^{2}$	**i.** $(64^{\frac{1}{2}}) \times 64$	**j.** $\frac{64^{\frac{3}{2}}}{64}$

3. Why do you think 64 was chosen for this task?

4. Practice explaining your answers to your partner and be prepared to speak to a bigger group or the whole class. You may find these key words helpful: ***exponent, power, root, reciprocal.***

EXTENSION QUESTIONS:

5. Choose a different number and repeat the task.

6. Replace 64 with *x* and write down any expressions that are always equivalent.

Ideas for Teaching Activity 2

- It might be helpful to discuss one expression with the whole class to begin with, perhaps asking for answers on show boards.
- It might help learners to have the cards (page 12) so that they can sort the cards and put side by side the cards that have the same value.
- Notice that this activity has built in **differentiation**; it is suitable for learners of different abilities. Slower learners should have success with some of questions 1 to 4 while the extension questions 5 and 6 provide a challenge for more able learners.
- After 10 -15 minutes of work The pairs could come together as groups of 4 learners to share their solutions, ask each other questions and note down anything they need further help with. These questions can then be given to the teacher.
- A whole class discussion at the end of the lesson could include:
    - A quick recap with show boards where learners given the values for each of the expressions.
    - Presentations by learners who explain the number they chose for question 5.
    - Presentations by learners who explain their generalisations for question 6.
- Every time learners make a contribution they should be asked to explain their reasoning and encouraged to use appropriate vocabulary.

Changes in my classroom practice

It is important that the intention of the lesson is shared with the learners. The first activity, making the larger triangle from the 16 small triangles, uses the prior knowledge that the lessons will build on and extend. It is chosen to remind the learners of what they already know and, at the same time, to enable the teacher to assess what the learners remember and understand. All the activities in this workshop guide are designed to review and extend earlier work on exponents and surds.

As the teacher you need to plan activities to support weaker learners when necessary and other activities to challenge high flyers. If the formative assessment shows that the majority of the class fail to remember and understand the earlier work then you may have to change your lesson plan and engage the whole class in the work you had prepared for the weaker learners. *Talking* about ideas, and *explaining* what they understand to other learners, as well as *listening* to other learners, deepens understanding. Once they are talking you can listen and try to understand their thinking and use what you hear to inform your teaching.

Discussion between learners, and giving reasons for their answers in class discussion, should draw out all the ideas that learners already have. These tasks should help learners to understand what they are doing when they manipulate equations and expressions with exponents and surds. The teacher needs to introduce the formal language and notation and to give the learners lots of opportunity, both in discussion and in writing, to use and understand it.

We know as teachers that explaining something to someone else is a real test of how well we understand it. Learners will need a lot of help and strong modelling from the teacher if they are going to move away from "*this is what you **do***" to "*I can explain what this **means***."

Learners should expect to find it difficult to explain their reasoning and they will have to work hard at this and help each other. The mathematical vocabulary and symbolism should make it easier to be precise and clear in a way that ordinary everyday language may not do and it should be regarded as a useful aid and not something to confuse us.

Activities such as sorting cards can help learners in their thinking. Resources can be re-used by different classes again and again if the cards are kept in envelopes and collected at the end of lessons.

Common errors and misconceptions

We have two ways of writing fractional exponents, for example cube roots as $x^{\frac{1}{3}}$ and $\sqrt[3]{x}$. This can be confusing for learners who might assume that they mean different things.

Learner need to appreciate that we write down mathematics according to what mathematicians have defined and agreed on as standard notation, for example $\sqrt[3]{x}$ for cube roots and $x^{-2} = \frac{1}{x^2}$ for the reciprocal of x^2. They also need to understand that different ways of writing the same thing can arise to make for consistency with the rules for positive integer exponents.

For example, by definition $\sqrt[3]{x}\times\sqrt[3]{x}\times\sqrt[3]{x}$ = x and $x^{\frac{1}{3}}\times x^{\frac{1}{3}}\times x^{\frac{1}{3}} = x^{(\frac{1}{3}+\frac{1}{3}+\frac{1}{3})} = x^1 = x$ so, for cube roots: $\sqrt[3]{x} = x^{\frac{1}{3}}$.

By definition $x^3 \div x^5 = x^3\times\frac{1}{x^5} = \frac{1}{x^2}$ so it follows that $\frac{1}{x^5} = x^{-5}$and $\frac{1}{x^2} = x^{-2}$.

For reasons of mathematical consistency mathematicians choose to define $x^0 \equiv 1$. If this causes confusion then it can be useful to list the powers of 10 from 10^4 to 10^{-4} that is 10^4, 10^3, 10^2…10^{-4} and to see that each time we reduce the power of 10 by 1 we are dividing by 10 so that the only logical value for 10^0 is 1.

It may be necessary to spend some time reviewing the simplification of surds, for example $\sqrt{8} = \sqrt{(4\times2)} = 2\sqrt{2}$ and discussing why $\sqrt{ab} = \sqrt{a}\times\sqrt{b}$ whereas $\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$.

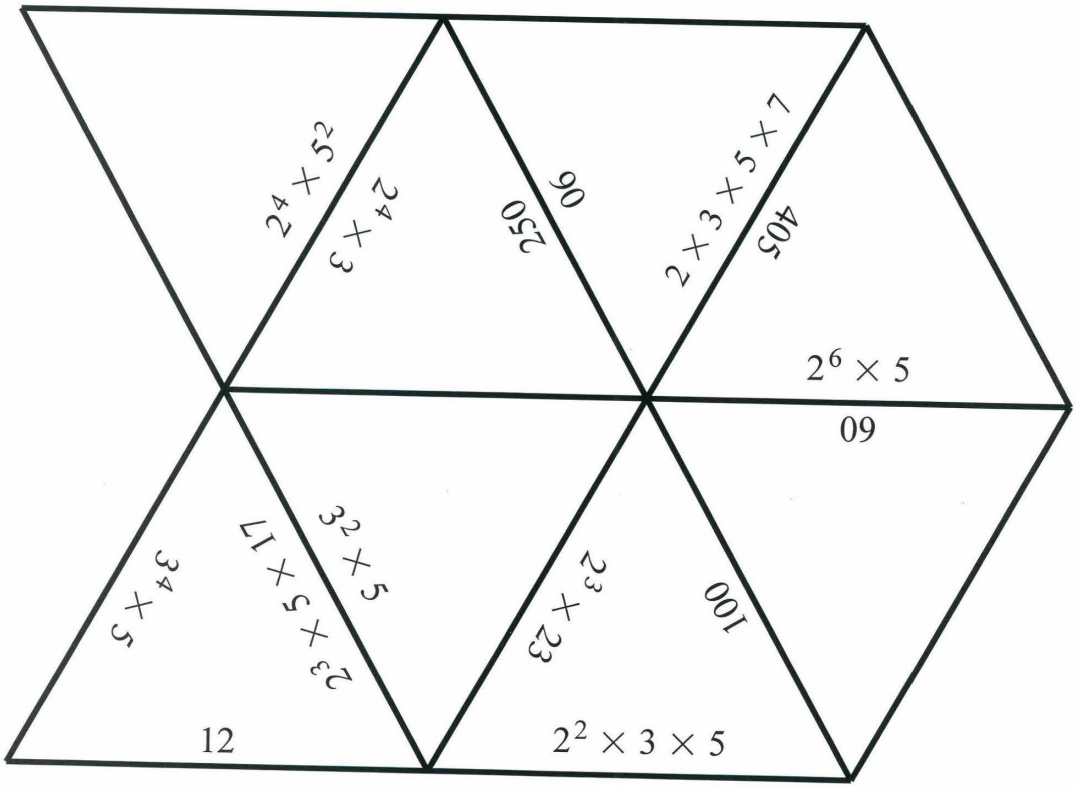
Card set 1 Explore 64

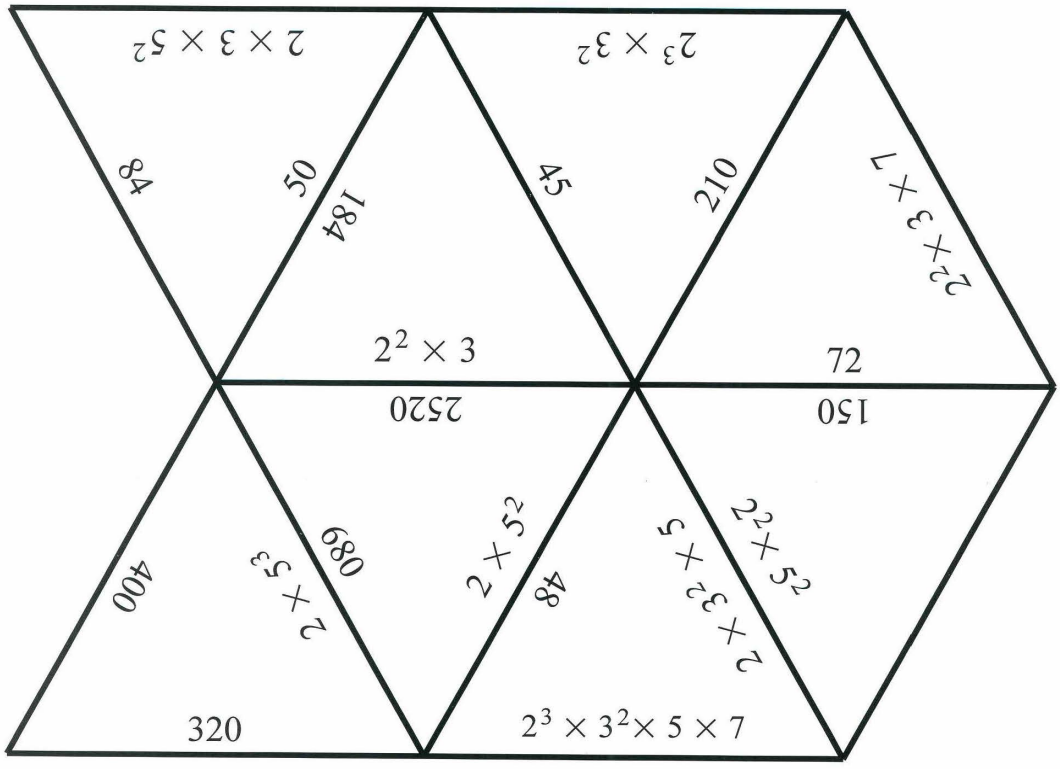
a. $64^{\frac{1}{2}}$	**b.** $64^{\frac{1}{3}}$
c. $64^{\frac{2}{3}}$	**d.** $64^{\frac{3}{2}}$
e. 64^0	**f.** 64^1
g. $(64^{\frac{1}{2}})^3$	**h.** $(64^{\frac{1}{3}})^2$
i. $(64^{\frac{1}{2}}) \times 64$	**j.** $\frac{64^{\frac{3}{2}}}{64}$

Card set 2

Power Matching Puzzle

(These 16 small triangles need to be cut out)





FOLLOW UP WORK

If you want to make your own jigsaws and puzzles you can download free software called "**Formulator Tarsia**" available from the Hermitech Laboratory website .

With Tarsia you can create, print out, save and exchange customised jigsaws, domino activities and a variety of rectangular card sort activities.

The following Inquiry Based Learning Activities are all on the AIMING HIGH Teacher Network.

Making Six https://aiminghigh.aimssec.ac.za/grades-7-to-9-making-six/

Easy Calc https://aiminghigh.aimssec.ac.za/grades-7-to-10-easy-calc/

Power Matching https://aiminghigh.aimssec.ac.za/grades-7-to-10-power-matching/
See this lesson activity for the solution to Classroom Activity 1 (page 4) and for more information and suggestions for planning a lesson to use this activity.

Powerful Thinking 1 https://aiminghigh.aimssec.ac.za/grade-7-powerful-thinking-1/

Powerful Thinking 2 https://aiminghigh.aimssec.ac.za/grades-8-10-powerful-thinking-2/

Powerful Thinking 3 https://aiminghigh.aimssec.ac.za/grades-7-to-9-powerful-thinking-3/

Powerful Thinking 4 https://aiminghigh.aimssec.ac.za/grades-10-powerful-thinking-4/

How Many Factors https://aiminghigh.aimssec.ac.za/grades-9-to-11-how_many_factors/

Exponents https://aiminghigh.aimssec.ac.za/grades-9-11-exponents/
This lesson activity is a formative assessment or revision task for Grade 9 or 10 learners that will enable the teacher to assess how much the learners know and understand about exponents before moving on to further work on the topic.