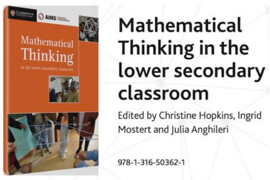
AIMSSEC

WORKHOP GUIDES FOR TEACHERS TO LEARN TOGETHER
UPPER SECONDARY GI SIMILARITY & ENLARGEMENT
Guide for your own self-help professional development workshop
and resources for inquiry based lessons.

MANAGE YOUR OWN PROFESSIONAL DEVELOPMENT WORKSHOP

These guides are designed to support teachers in developing a deep understanding of the mathematics they teach and in developing more effective ways of teaching.

You can use these guides on your own or as one of a group of teachers who meet together to talk about your mathematics lessons as part of your professional development. Maybe one of you will take the lead in organizing time, date and venue but once you are doing the activities together you will all participate on equal terms in the discussion and reflection.



The AIMSSEC Mathematical Thinking Book provides similar workshop guides.
Buy the book online from Amazon or http://www.cambridge.org/za/education
Search for AIMSSEC or for ISBN 9781316503621.
To order the book in South Africa go directly to http://www.cup.co.za
Reviews & curriculum map: https://aiminghigh.aimssec.ac.za/mathematical-thinking/

EACH WORKSHOP GUIDE HAS A SIMILAR FORMAT:

PAGE 1 **TITLE PAGE**
Teaching strategy.
Curriculum content and learning outcomes.
Summary of mathematical topic (FACT BOX.)

PAGES 2 & 3 WORKSHOP ACTIVITIES FOR TEACHERS
Two pages for you to work through with your colleagues. These are activities to be shared and discussed. For each activity there is a list of resources needed , how to organise the activity (e.g. individual, pairs, whole class) , how long the activity will take , when to pause, think and try the activity , and when to record your work .

PAGES 4 & 5 CLASSROOM ACTIVITIES FOR LEARNERS
Two pages to help you plan your lesson. You are advised how long to allow for the activity, the resources you might need and the key questions to ask.

PAGES 6 TO 10 **CHANGES IN MY CLASSROOM PRACTICE**
Pages on implementing the teaching strategies with additional resources and activities for use during or after the workshop such as worksheets and templates.

UPPER SECONDARY GEOMETRY 1 SIMILARITY AND ENLARGEMENT
by Toni Beardon

Toni has taught for 55 years in schools and universities in the UK, USA and South Africa. At Cambridge University in 1987 she established a new postgraduate teacher training programme and a student community service project. In 1996 she set up the NRICH Online Mathematics Club and other online projects, now the Cambridge University Millennium Mathematics Project. In 2002 Toni founded AIMSSEC. Now retired, Toni divides her time between Cambridge and Cape Town working to provide better educational opportunities for children from disadvantaged communities and to improve the standard of mathematics teaching and learning in Africa in all types of schools.

Similarity and Enlargement

Teaching Strategy: Learning through investigation and discussion

Curriculum content: Properties of congruent and similar figures. Use of geometry of straight lines and triangles to solve problems and justify relationships in geometric figures. Use of proportion and scale factors to describe the effect of enlargement and reduction on properties of geometric figures.

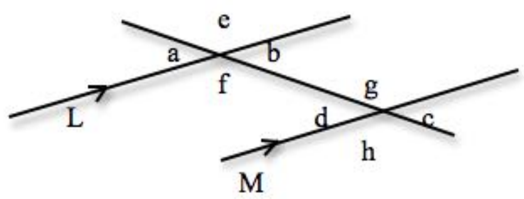
Prior knowledge: Angle relationships for a transversal intersecting parallel lines. Vertically opposite, alternate and corresponding angles. Learners can find area by counting squares and volume by counting cubes.

Intended Learning Outcomes: At the end of this activity teachers and learners will:

- ✓ Know that enlargements produce similar figures where the corresponding angles are equal. Know that if the linear scale factor of an enlargement is k then the area scale factor is k^2 and the volume scale factor is k^3.
- ✓ Understand a proof that the angles of a triangle add up to 180^o.
- ✓ Be able to use appropriate vocabulary in simple geometric arguments.
- ✓ Appreciate the value of working on your own and also of sharing ideas.
- ✓ Have experienced being able to learn and make mathematical discoveries for yourself.

Fact Box

Vertically opposite angles: a=b, e=f, c=d, g=h

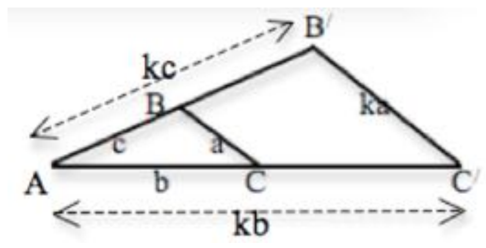


If L and M are **parallel lines** then the following angles are equal:
Alternate angles: b=d, f=g
Corresponding angles: a=d, b=c, e=g, f=h
Conversely: If one of these pairs of angles are equal then the lines L and M are parallel.

Similar shapes have the same shape and angles. Congruent shapes are the same in all respects.

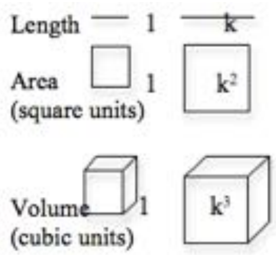
A tessellation is a pattern of congruent plane shapes that fills the plane with no gaps and no overlaps

Similar triangles have corresponding angles equal.



ΔABC has edges a, b and c.
If $\angle ABC=\angle AB'C'$ then these are corresponding angles so
BC is parallel to $B'C'$ and
ΔABC is similar to $\Delta AB'C'$.
$\Delta AB'C'$ is an enlargement of ΔABC with centre A and scale factor k.

Scale factors

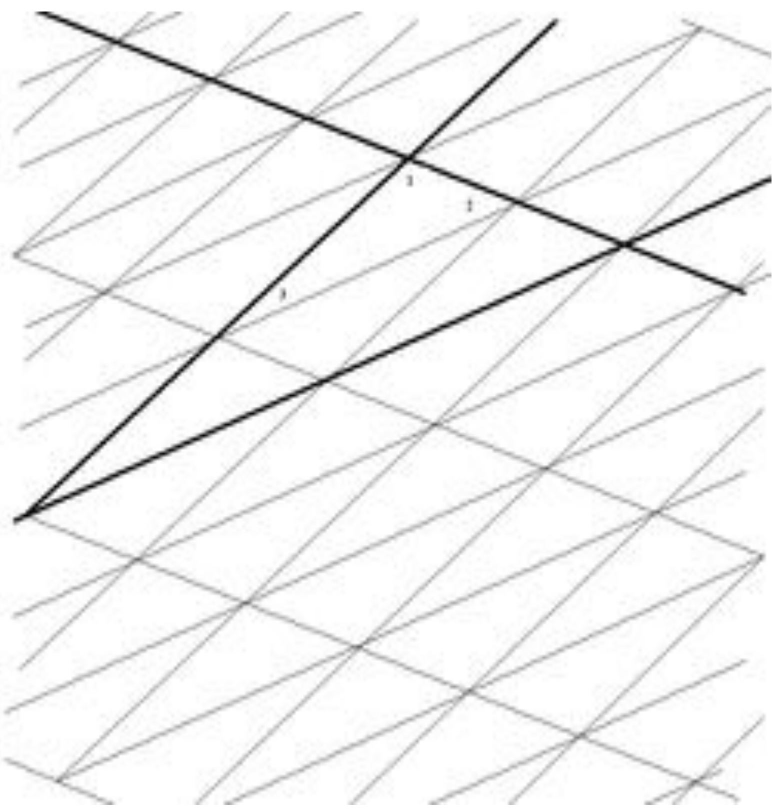


If the **linear scale factor** of an enlargement is k
the **area scale factor** is k^2
and the **volume scale factor** is k^3.

Workshop Activities for Teachers

Activity 1 Teasing Tessellation

Ruler, protractor, notebook, coloured pens. 45 minutes



First work on your own for at least five minutes to write down answers to the five questions below.

You must give reasons for your answers.

Then pair up and share your ideas with a partner for ten to fifteen minutes.

Then make foursomes and share your ideas.

Did you and your partner notice exactly the same things as the other pair?

When you are ready have a discussion with the whole group and make a list of ALL the ideas that have come up.

Assuming that there are three sets of parallel lines in the diagram:

1. What do you notice in this diagram?
2. What shapes do you see?
3. Mark the angles equal to angle 1.
   Mark the angles equal to angle 2.
   Mark the angles equal to angle 3.
4. What properties of the shapes can you deduce from the diagram?

What geometrical facts can you prove from this diagram?

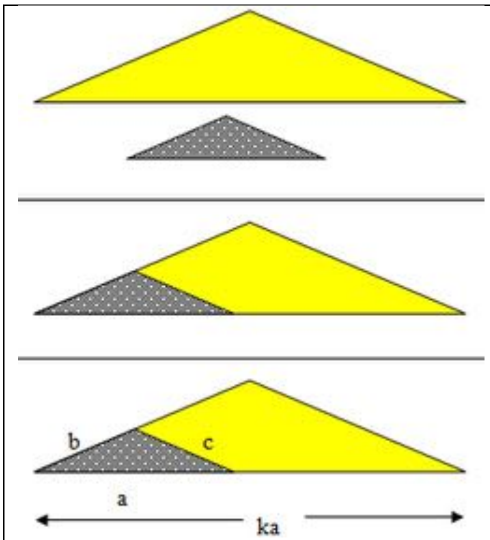
Notes to help you do the activity

1. By working alone, then in a pair, then a foursome and finally sharing ideas with the whole class you should be able to make a long list of observations about this diagram.
2. Use colour for labeling equal angles and equal lengths to help identify the equal elements.
3. Using vertically opposite, alternate and corresponding angles it is possible to identify every angle as equal to angle 1 or angle 2 or angle 3.
4. The pattern is a tessellation of congruent triangles. Find enlargements of the small triangle. Put two or more triangles together, find 3 different sets of parallelograms (not rhombuses), also trapezia and hexagons.
5. In the same way a tessellation pattern can be drawn for any triangle. Observe that angles 1, 2 and 3 form three adjacent angles on a straight line and use this to prove that the angles of any triangle add up to 180^o.
6. You will find many geometrical properties and relationships in this diagram. Here are a few:

- Can you see rotations of 180^o that map the pattern into itself?
- Can you see that there are no reflections in this pattern?
- Study the triangle with the heavy outline. It has sides twice as long as the small triangle and it contains 4 copies of the original triangle.
- Can you see, and explain why, the lines joining the midpoints of the edges are parallel to the third side of this triangle and half its length (this is the Triangle Midpoint Theorem)?

7. See Tessellating Triangles https://aiminghigh.aimssec.ac.za/grades-7-to-12-tessellating-triangles/

Activity 2 Similar Triangles 15 minutes



k is the **scale factor** of the enlargement

Two triangles that have the same angles are called similar triangles.

Cut out two identical scalene triangles and then make one of them smaller by cutting along a line parallel to one of the edges. They can be rotated and reflected to place them so that corresponding edges in the triangles are parallel.

Check that the angles in one of your triangles are equal to the angles in the other triangle.

Measure the lengths of the edges and check that your triangles have sides in proportion, say a, b, c and ka, kb, kc.

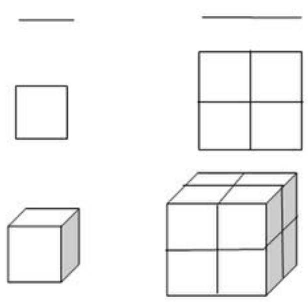
Make a vertex in each triangle coincide. Place sides a and ka along the same line.

If you construct a triangle on the base of length ka with sides kb and kc then there are only two possible triangles – the one shown in the diagram and its reflection in the line ka.

This demonstrates that triangles with sides in proportion are similar. Relate this discussion to the diagram for Activity 1.

Activity 3 Scale Factors

Small cubes (at least 27) 15 minutes



Consider the enlargements shown in these diagrams.

Draw similar illustrations for enlargements with linear scale factors 3 and 4. How many squares and cubes are there in your pictures?

Use small cubes to model this if available.

If the **linear scale factor is k**

then the **area scale factor is k^2** (dimension 2)

and the **volume scale factor is k^3** (dimension 3)

Note: the relationship between linear, area and volume scale factors in an enlargement is a fundamental and important concept .

Follow Up Activities

Working in groups you can use a torch, or sunlight, to produce a shadow of shapes on the wall, taking care that the shadow is an enlargement and not a distorted image. Someone can measure the dimensions of the image. You can draw the enlargement on newspaper and work out approximate areas for the object and its enlarged image. Teachers can use the opportunity to introduce a discussion of why scale is important (films, photos, maps, architectural plans etc.) and all the ways we experience enlargements in everyday life.

Other follow up activities could be the learning activities:
Simsets https://aiminghigh.aimssec.ac.za/grades-8-to-10-simsets/
Is a square a rectangle? https://aiminghigh.aimssec.ac.za/grades-5-to-10-is-a-square-a-rectangle/

Classroom Activities for Learners

Activity 1 Reptiles (Repeating tiles)

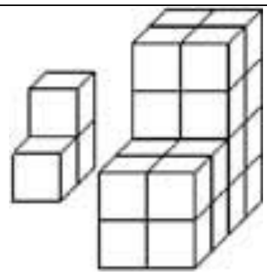
Resources: Squared paper. Cubes are useful but optional. **Time:** 30 minutes**.**

<table>
<tr>
<td rowspan="2"></td>
<td rowspan="2">Notice that the four shapes in each set are congruent and they are all made up of small squares.

On squared paper draw each of the three shapes and find their areas.

Draw an enlargement of each shape such that four identical smaller shapes fit together to make the enlarged shape

How many times longer are the edges (the linear scale factor of the enlargement)?

How many times bigger is the area (the area scale factor of the enlargement)?</td>
<td></td>
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<tr>
<td>Now look at the illustration of the three dimensional solids. One is an enlargement of the other. Make the shapes with cubes.

What is the linear scale factor of the enlargement?

How many times bigger is the surface area (the area scale factor)?

How many times bigger is the volume (the volume scale factor)?</td>
</tr>
</table>

Ideas for Teaching

1. This example is a puzzle in which the learners have to fit the shapes together in different ways until they make an enlargement of the original. Using squared paper they can make drawings and they will need to experiment systematically until they find solutions. Cutting out the four similar shapes and handling them is helpful, particularly for young learners but they should be able to find the solutions by drawing.

2. You might want to use the ‘one, two, four, more’ strategy just for the first puzzle so that everyone in the class fully understands what they are supposed to do. For the other three parts of the question learners should work singly because it is important to give them every opportunity to discover the solutions for themselves.

3. Knowing that the area scale factor is four it follows that the linear scale factor is two which determines the lengths of the edges (and the outline) of the enlarged shape. The learners should be able to discover this for themselves so the teacher should not tell them this but only give guidance by asking key questions.

4. The activities provide material that teachers can use to establish a culture in lessons where the learners experience guided re-invention. These activities are open ended and many geometrical properties can be identified and proved. The diagram from the Teachers’ Workshop Activity 1 can be given to learners as a sort of ‘treasure hunt’: “What can you find in this picture?” either to review earlier work or to lead to new. Learners can label the diagram themselves and present their findings, either on the chalkboard or on posters, making suggestions that the class can discuss.

5. See the Notes for Teachers on https://aiminghigh.aimssec.ac.za/grade-9-or-10-enlargement/

Classroom Activities for Learners

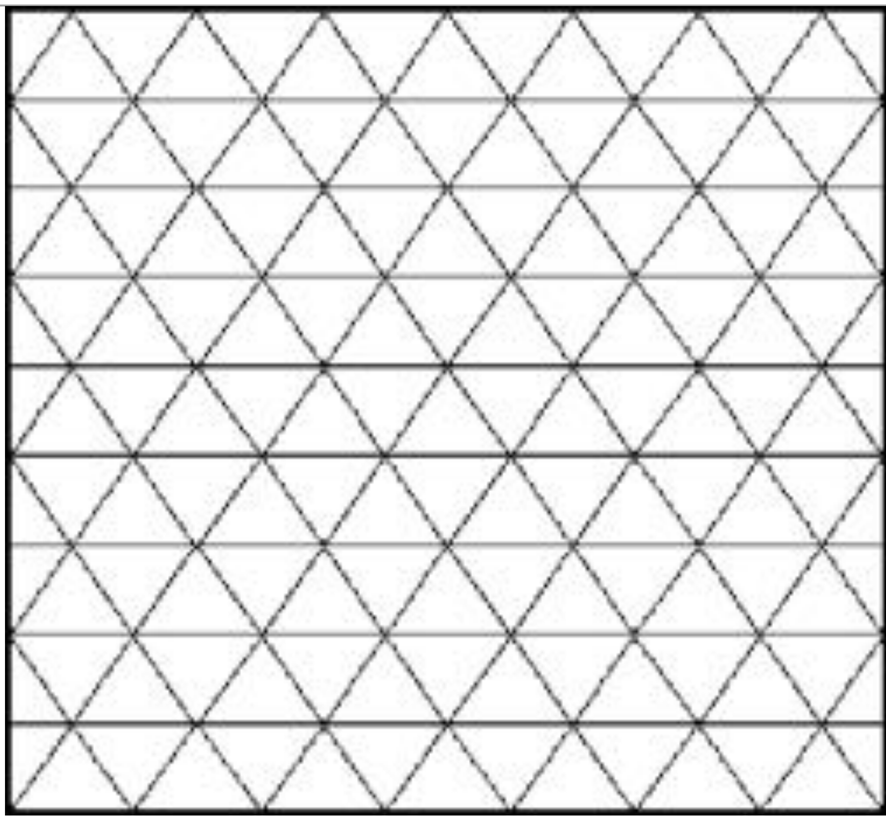
Activity 2: Where are they?

Isometric paper as shown here or download from http://nrich.maths.org/6676 20 minutes

Use isometric grid paper (or a dotted isometric grid) to find and draw the following polygons with vertices on the lattice points of the grid.

- A rectangle
- A rhombus
- A trapezium
- A parallelogram that is not a rectangle
- An equilateral triangle
- A right angled triangle
- A scalene triangle
- An isosceles triangle that is not an equilateral triangle
- A pentagon
- A hexagon
- A heptagon
- An octagon

Can you find enlargements of some of the shapes you have drawn?



Reference: See Where are They? www.nrich.maths.org/1058 and http://www.mathleague.com/

Ideas for teaching

This activity can be introduced to the whole class who then work in pairs. Tell the learners that many different shapes are hidden, and ask if they can see any. Draw an equilateral triangle. Ask what it is called and invite learners to draw another equilateral triangle of a different size. Explain that they don't have to go over the lines of the grid but they must join up the vertices.

Introduce the name of each of the shapes in turn, share understandings of the properties of each shape and allow some time for the learners to work in pairs to see how many different shapes they can find. Discuss with the class after each two or three, and invite them to illustrate for all to see. Does everyone agree? Encourage the learners to name their shapes on their grid.

Activity 3 Teasing Tessellation

The class can do Activity 1 from the Teacher Workshop with the grid of a tessellation of scalene triangles. They can first look for shapes as in the 'Where are they?' activity then go on to investigate and prove geometrical properties and relationships.

Discussion of Teaching Strategy

In the Teacher Workshop Activity 1 teachers have experienced the teaching strategy called 'one-two-four-more' or 'think, pair, share'. It works well for very large classes where there is no room for any other sort of group work. This strategy gives the maximum chance for many different ideas to emerge and learners may even think of methods and ideas that the teacher had not thought of. This is a useful teaching strategy for many lessons.

- **One**: At the start of the lesson, introduce a task and ask learners to spend a few minutes 'doing' the task on their own so that everyone is given time to think individually. The task should be such that everyone can make a start. This stage is important so that nobody can get the idea that they just wait for someone else to do the work that is one of the problems that can arise with group work in class. Working individually at the start gives everyone time and space to read the introduction to the task and to think quietly so as to sort out their own ideas.
- **Two**: Then, at a signal from the teacher, the learners work in pairs and 'talk' about the task. They should discuss the methods they have been using. Is anyone stuck? Can they help each other? Are they thinking the same way? Did they think of the same method or different ways of tackling the task? What can they learn from each other? They should do the task together by at least one method, record their work and check their answers. This paired work should take up at least half the lesson. Even in the largest class, with limited space, it is usually possible to organize paired work.
- **Four**: When the teacher decides that the moment is right she instructs each pair of learners to work with another pair to make up a foursome. They may find that they have taken different approaches and have valuable ideas to exchange. For smooth transitions the teacher may decide that the learners should sit in the same places in all the mathematics lessons, work with the same partners and make up the same groups of four. For example if the class are not sitting around tables half the pairs might turn around to work with the pair behind them. They can compare methods and check to see if they have got the same answers or equivalent answers. They might try to do the task by an alternative method and record their work. Finally each foursome will have something worthwhile to contribute to a whole group discussion.
- **More**: This is the whole class discussion at the end of the lesson when the teacher builds on what the learners have discovered for themselves. The teacher calls on representatives of different groups to report to the whole class on how they tackled the task and what they have found out, then summarizes the ideas and perhaps adds some explanation, for example on the connections between the different approaches used or the different points of view taken. The teacher's role is to endorse and summarize the suggestions, eliciting or giving explanations, or alternatively when necessary to help the learners to put right any mistaken ideas in the spirit of learning from mistakes. Learners won't find it easy if they are unaccustomed to the discovery way of learning so the teacher will need to refrain from telling them what to do but give them plenty of encouragement and praise when they make observations. The final summary and recording of results is an essential stage in the process and should not be rushed.

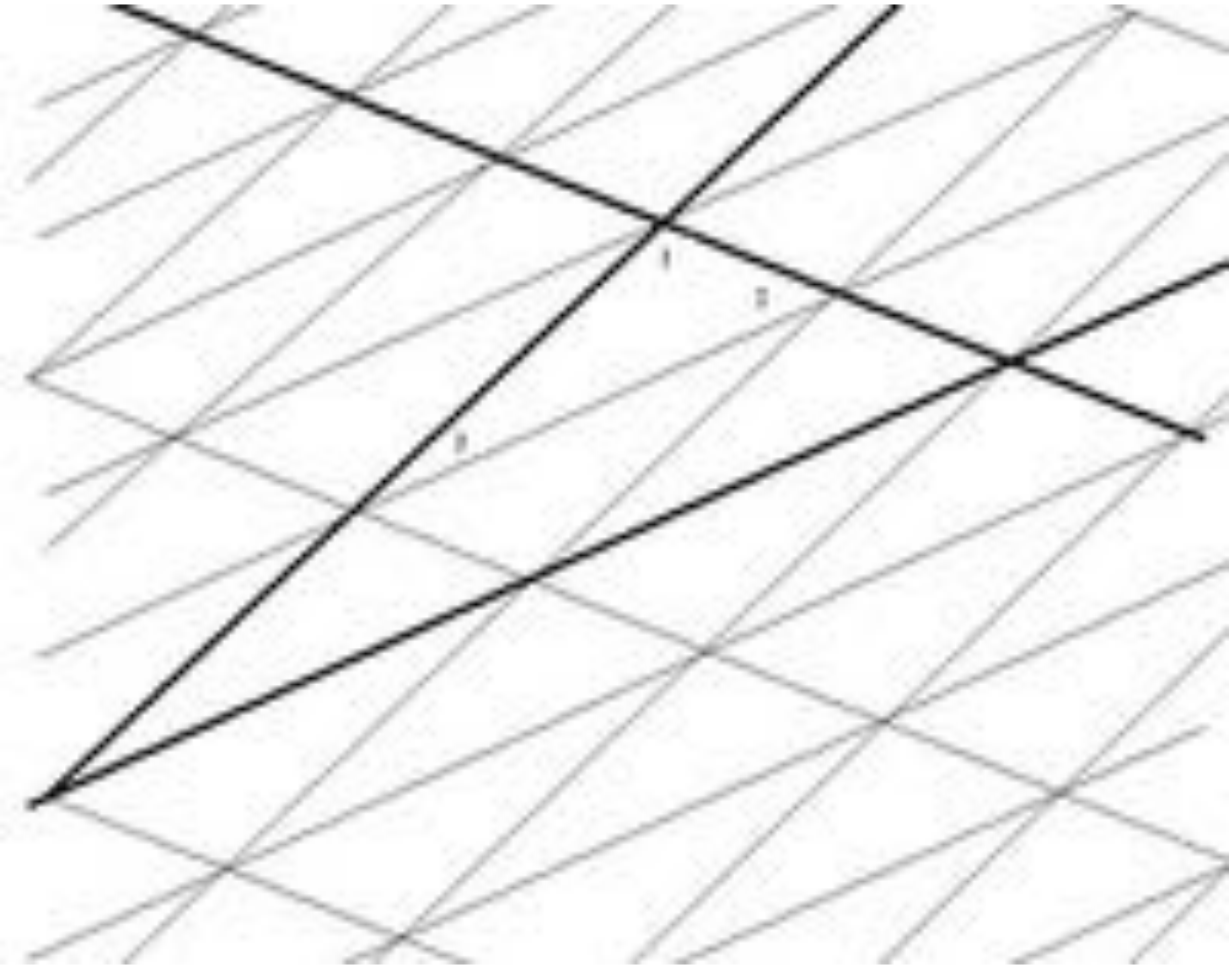
Further reading:

Thinking tools bring sound instructional theory into the classroom in a practical form that students and teachers both enjoy using. Jay Mctighe and Frank T. Lyman
http://ascd.com/ASCD/pdf/journals/ed_lead/el_198804_mctighe.pdf

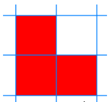
Key Questions (All answers require reasons):

In this diagram there are 3 sets of parallel lines.

1. On the diagram mark angles equal to angle 1, angle 2 or angle 3.
2. Which shapes are congruent?
3. Which lengths in the diagram are equal?



4. What transformations map shapes in the diagram to congruent shapes.
5. Which shapes are similar? What is the scale factor?
6. If the area scale factor is *x* what is the linear scale factor?
7. Explain how you can prove from the diagram that the angles of a triangle add up to 180°.
8. Explain how you can prove from the diagram that the diagonals of a parallelogram bisect each other.
9. Explain how you can prove from the diagram that the line joining the midpoints of two sides of a triangle is parallel to the third side and half the length of the third side (Triangle Midpoint Theorem).



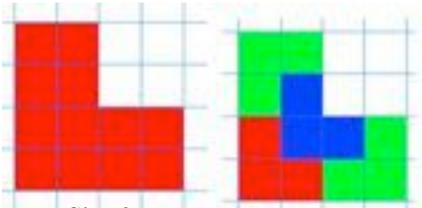
Size 1

TRISQUARES – An Extension Activity

Trisquares (also called tri-ominoes) are made up of three squares and each has an area of 3 square units. Can you fit them together to make an enlargement of the shape?

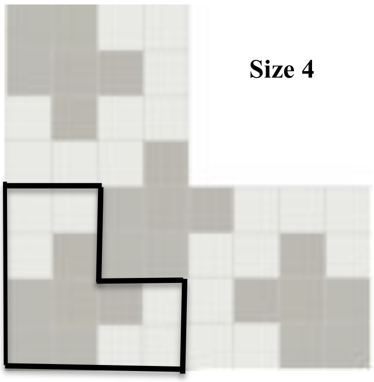
What is its area? Can you fit trisquares together to make enlargements of scale factors 3, 4 and 5? What are their areas? Is it possible to make enlargements of all sizes by fitting trisquares together?

Solution



Size 2

The diagram shows how 4 trisquares fit together to make a trisquare **enlarged** by a **linear scale factor** 2.
Comparing the 2 shapes, all the angles are the same and each edge in the bigger shape is twice the length of the corresponding edge in the smaller shape so the shapes are **similar**.
The **area scale factor** is 4. The areas are 3 squares and 12 squares. It might help to refer to them as 3-square and 12-square pieces.



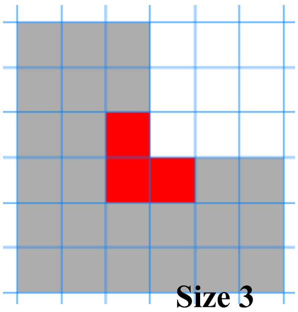
Size 4

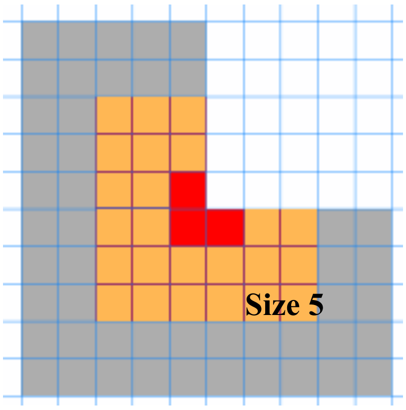
This is to help you to make a similar shape with area scale factor 16 using the tiling of 4 of the 12-square pieces. You might find it easier to do this next rather than to do the shape with linear scale factor 3.

Remember:

Basic shape	Area 3 squares	
Linear scale factor 2	Area 12 squares	Area scale factor $2^2 = 4$
Linear scale factor 3	Area 27 squares	Area scale factor $3^2 = 9$
Linear scale factor 4	Area 48 squares	Area scale factor $4^2 = 16$
Linear scale factor 5	Area 75 squares	Area scale factor $5^2 = 25$ and so on.

Can you see how to use 4 of the 12-square pieces outlined in the diagram above to make a similar shape (an enlargement) of area scale factor 16?
In this way the trisquares fit together to make enlargements of linear scale factors: 2, 4, 8, 16, 32 etc., that is corresponding to area scale factors: 4, 4^2=16, 4^3=64, 4^4= 256, 4^5=1024 and so on.





What about odd sized trisquares, for example linear scale factor 3 and area scale factor 9?
To make the enlargement by linear scale factor 3 shown, use one 3-square piece, one 12-square piece and two 2x3 rectangular blocks (each made up of two 3-square pieces). Try it for yourself.
To make the enlargement by linear scale factor 5 you simply add another 12-square piece in the bottom left and for the rest you can use the 2x3 rectangular blocks. Try it and see.

It's the same with the enlargement by linear scale factor 7, add a 12-square piece in the bottom left of the 75-square, size 5 piece and then use 2x3 blocks to tile the rest. This works with all odd numbers. There is a little more work to do to show that you can also make enlargements of linear scale factors 6, 10, 12 etc (that is even but not a power of 4) so that enlargements of all linear scale factors can be made just by fitting together the basic 3-square.
See https://nrich.maths.org/7026 for an interactive version of this activity and solutions from learners
Also see https://aiminghigh.aimssec.ac.za/grades-9-to-12-trisquares/