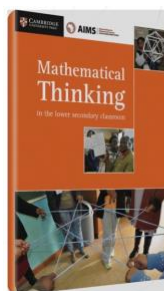


MANAGE YOUR OWN PROFESSIONAL DEVELOPMENT WORKSHOP

These guides are designed to support teachers in developing a deep understanding of the mathematics they teach and in developing more effective ways of teaching.

You can use these guides on your own or as one of a group of teachers who meet together to talk about your mathematics lessons as part of your professional development. Maybe one of you will take the lead in organizing time, date and venue but once you are doing the activities together you will all participate on equal terms in the discussion and reflection.



Mathematical Thinking in the lower secondary classroom

Edited by Christine Hopkins, Ingrid Mostert and Julia Anghileri

978-1-316-50362-1

These Lower Secondary Workshop Guides are chapters in the AIMSSEC Mathematical Thinking Book.

Buy the book online from [Amazon](http://www.amazon.com) or from <http://www.cambridge.org/za/education> Search for AIMSSEC or for ISBN 9781316503621. To order the book in South Africa go directly to <http://www.cup.co.za>

For reviews and curriculum map see <https://aiminghigh.aimssec.ac.za/mathematical-thinking/>



Let's Play Mathematically AIMING HIGH Family Games

By Toni Beardon and Caroline Ainslie

ISBN-978-1911093206

Buy the book or the e-book from the publisher [Tarquin](http://www.tarquin.co.uk) or from [Amazon](http://www.amazon.com). Having fun while you learn is the basis of this book. Over 50 games are provided with school curriculum related maths embedded in them. All that is needed to play the game can be downloaded.

The teachers' book **Let's Learn Mathematics Through Play** will be published in May 2026, full of ideas to help teachers in schools, and parents at home, to provide playful learning for children. Teachers who adopt playful learning in their teaching repertoire find that their students' attitudes to mathematics change, fear of the subject disappears, they remember and understand better, mathematical methods, number concepts and facts and they develop concentration and skills like computational thinking, logical reasoning and systematic planning.

The AIMSSEC App can be downloaded from the internet onto any android smart phone, laptop or tablet in 3 or 4 minutes and it is free. Go to Google Play and search for **AIMSSEC** and follow instructions. **Please register on the AIMSSEC App so that**, when you are connected to the internet, you can add comments, ask questions and join in professional discussions on the AIMING HIGH Teacher Network website.

After downloading the AIMSSEC App, everything will be available on your own phone or other device so that you will be able to use all the AIMING HIGH resources, lesson activities and professional development workshop guides offline, that is **WITHOUT USING THE INTERNET**. From time to time when you connect to the internet you can update your AIMSSEC App content with recently added resources.






All the resources on the **AIMING HIGH Teacher Network** are published under a **Creative Commons** Copyright License. This means that you can copy, adapt and use any AIMING HIGH material, as long as you state in print on your document that the source is AIMSSEC, except that you are not allowed to use the AIMSSEC material for profit.

EACH WORKSHOP GUIDE HAS A SIMILAR FORMAT:

PAGE 1 TITLE PAGE

- Teaching strategy focus. *Each guide focuses on and exemplifies a widely used teaching methodology.*
- Curriculum content and learning outcomes.
- Summary of mathematical topic (FACT BOX.)
- Resources needed

PAGES 2 & 3 WORKSHOP ACTIVITIES FOR TEACHERS

Two pages for you to work through with your colleagues. These activities are to be shared and discussed. For each activity there is a list of resources needed , how to organise the activity (e.g. individual, pairs, whole class) , how long the activity will take , when to pause, think and try the activity , and when to record your work .

PAGES 4 & 5 CLASSROOM ACTIVITIES FOR LEARNERS

Two pages to help you plan your lesson. You are advised how long to allow for the activity, the resources you might need and the key questions to ask.

PAGES 6 - 10 CHANGES IN MY CLASSROOM PRACTICE

Pages on using the teaching strategies with additional resources and activities for use during or after the workshop such as worksheets and templates. For follow-up activities you will find lots more lesson activities on the AIMING HIGH Teacher Network
<https://aiminghigh.aimssec.ac.za/category/lesson-activities/>

TRANSFORMATION OF FUNCTIONS

Proof of Pythagoras Theorem and its generalization to the Cosine Formula.

by Toni Beardon



Toni has taught for over 60 years in schools and universities in the UK, USA and South Africa.

At Cambridge University in 1987 she established a new postgraduate teacher training programme and a student community service project. In 1996 she set up the NRICH Online Mathematics Club and other online projects, now the Cambridge University Millennium Mathematics Project (MMP).

In 2002 Toni founded AIMSSEC to provide better educational opportunities for children from disadvantaged communities and to improve the standard of mathematics teaching and learning in Africa in all types of schools.

Toni has written Let's Play Mathematically and Let's Learn Mathematics Through Play to provide teaching and learning resources to enrich student's experience of learning.

Workshop for Teachers and Teaching Resources on the Proof of Pythagoras Theorem and its generalization to the Cosine Formula.

Teaching Strategy: Developing Mathematical Reasoning and Proof

Curriculum content: Proof of Pythagoras Theorem by similarity. Proof of Cosine Formula.

Prior knowledge: Understand that triangles that have the same angles are similar and the ratios of corresponding sides are equal. Similar triangles can be aligned so one is an enlargement of the other and the ratio of corresponding sides is the scale factor of the enlargement. Be able to calculate the areas of right angled triangles and squares.



The earliest record of the theorem is a Babylonian tablet about 3000 years old.

Intended Learning Outcomes: At the end of this activity teachers and learners will:

- ✓ Know the logical difference between a theorem and its converse.
- ✓ Understand how to prove Pythagoras Theorem using similarity.
- ✓ Be able apply the theorem in problems in two and three dimensions.
- ✓ Appreciate that proof is essential to mathematics.
- ✓ Have experienced deriving formulas and developing proofs for themselves.

Fact box

Pythagoras Theorem:

If a triangle is right angled then the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides. If the sides have lengths a , b and c then $a^2 + b^2 = c^2$.

Converse of Pythagoras Theorem:

If the area of the square on the longest side of a triangle is equal to the sum of the areas of the squares on the other two sides then the triangle is right angled.

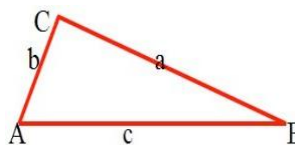
Cosine Formula

$$a^2 = b^2 + c^2 - 2bccosA$$

$$b^2 = c^2 + a^2 - 2cacosB$$

$$c^2 = a^2 + b^2 - 2abcosC$$

$$cos A = (b^2 + c^2 - a^2) / 2cb$$



The Cosine Formula is a generalisation of Pythagoras Theorem or alternatively Pythagoras Theorem is a special case of the Cosine Formula.

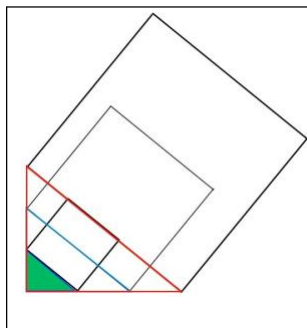
Workshop Activities for teachers

Activity 1 Warm-up

5 minutes

Organisation: Work in pairs. Every 10 minutes discuss your findings with the whole group.

Resources: A board or flip chart for sharing ideas.




Discuss the similarities, enlargements and scale factors that you see in the diagram.

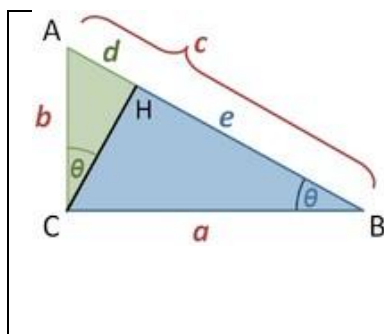
Note that the triangles are **not** isosceles triangles.

The perpendicular edges of the large triangle are split into three equal lengths.

Can the statements you make about this diagram be generalised to apply to enlargements for any scale factors?

 Try this now

Activity 2 Proof of Pythagoras Theorem




Angle ACB is a right angle and CH is perpendicular to AB.

Prove that $\angle ACH = \angle ABC$, identify the similar triangles in this diagram and prove that they are similar.

Use similarity to give d in terms of b and c

and then to give e in terms of a and c.

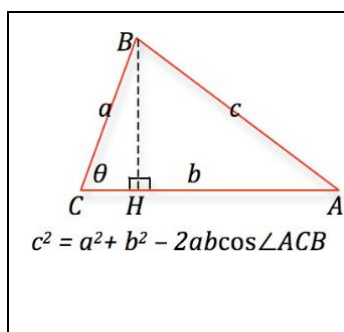
What can you deduce from these results?

 Try this now

Notes to help you do the activity.

1. Don't spend long on Activity 1. It is just to get you thinking about similarity and enlargement.
2. By asking questions about the diagram in Activity 2 teachers can guide learners to derive the proof of Pythagoras Theorem by similar triangles for themselves.
3. By asking questions in Activity 3 teachers can guide learners to derive the proof of the Cosine Formula for themselves.
4. You need to think clearly about the difference between a theorem 'we know P is true so it follows that Q is true' (given for some statements P and Q) and its converse 'we know Q is true so it follows that P is true'. In the case of Pythagoras Theorem both theorem and converse are true but with other theorems this is not always the case.
5. By asking questions in Activity 4 teachers can guide learners to derive the proof of the Converse of Pythagoras Theorem for themselves.

Activity 3 Cosine Formula




In any triangle ABC, let H be the foot of the perpendicular from B to AC. Now think about the right angled triangle BHC.

Show $BH = a \sin \theta$, $CH = a \cos \theta$ and $AH = b - a \cos \theta$.

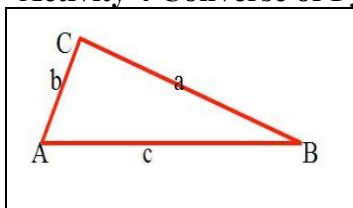
Now use Pythagoras Theorem for triangle AHB to prove the Cosine Formula for triangle ABC.

What does the cosine formula become when $\angle ACB = 90^\circ$?


Discuss how the cosine formula gives a generalisation of Pythagoras Theorem.

 Try this now

Activity 4 Converse of Pythagoras Theorem

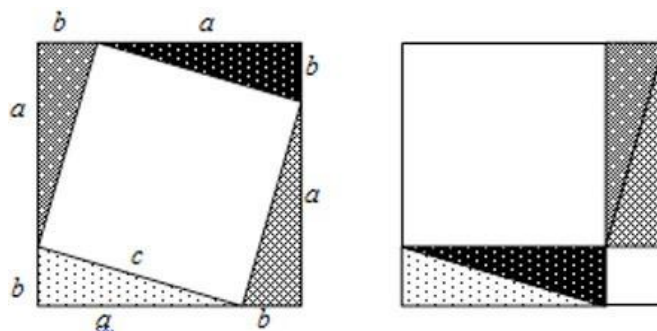


You have proved the cosine formula for any triangle ABC, that is:
 $c^2 = a^2 + b^2 - 2ab \cos \angle ACB$

Suppose you don't know $\angle ACB$ but you do know that $c^2 = a^2 + b^2$.
 In this case what does the cosine formula tell you about $\angle ACB$?
 This is the converse of Pythagoras Theorem.  Try this now

Discussion of pedagogical issues:

1. Discovery and proof are essential elements in learning mathematics.
2. When teachers first introduce Pythagoras Theorem, they often use the diagram shown on the right to give learners the opportunity to use their knowledge of areas to discover the theorem for themselves.
3. Four **identical (congruent) right angled triangles** are arranged in the squares shown in the diagrams which have sides $(a + b)$. Think about the areas in the diagrams. Just take away the four identical right angled triangles from each of the diagrams. What is left? What does this tell you about the areas of the unshaded parts of the diagrams? Write this as a formula involving a , b and c .
4. To introduce Pythagoras Theorem by telling learners the formula is not good teaching. It robs learners of the opportunity to think mathematically and to understand where the formula comes from. Mathematics should always be taught so that learners understand how to derive the formulas, and why the methods work, even if the proofs are not assessed by examination.
5. Always be on the lookout for connections. It is important to recognise that the Cosine Formula is a generalisation of Pythagoras Theorem.
6. The proof of the converse of Pythagoras Theorem (Activity 4) follows simply from the proof of the Cosine Formula.

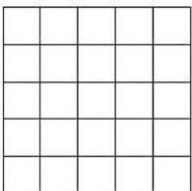
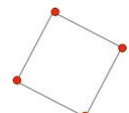


References: See the interactivity Matter of Scale <http://nrich.maths.org/811> and <http://nrich.maths.org/6553> . For more proofs of Pythagoras Theorem, the wonderful website Cut the Knot has 88 versions <http://www.cut-the-knot.org/pythagoras/>

Classroom Activities for Learners

Resources: Scrap paper, ruler, scissors, calculators.

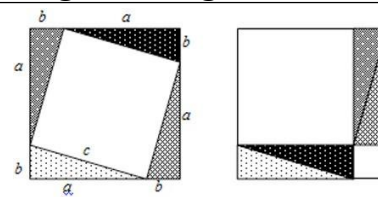
Activity 1 Squares Game - warm up 5 minutes, can be used often

	<p>How to play the game: This is a game for 2 players or two teams. Take it in turn to mark one intersection point with your colour (or special mark) on a grid like the one on the left. Try to make a square with your own marks at the four corners. Try to stop your opponent making a square. Don't forget the tilted squares. The winner is the first player to make a square.</p>	
---	---	---

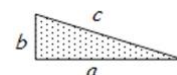
Activity 2 Investigation of Squares Built on the Sides of a Right Angled Triangle 1 hour

Look at the two diagrams on the right. All the triangles are congruent. If you take away the 8 copies of the triangle, what is left behind? Think about areas? Discuss what you see and what you think it means.

To investigate this further and check all the ideas, cut out 4 identical right angled triangles. Each learner in the class will probably have a different triangle.



1. Measure the three sides a , b and c in centimetres correct to the nearest millimetre (e.g. 5.2 cm) and write down your measurements.
2. Arrange your triangles like the diagram on the left above. What shape is the quadrilateral in the middle surrounded by the four triangles? How can you be sure?
3. What is its area in terms of c ? Calculate the area for your own triangle.
4. From your measurements add up $(a+b)$. Draw a square with sides of length $(a+b)$. Arrange your 4 triangles in the square you have drawn like the diagram on the right.
5. How can you be sure that the two uncovered (unshaded) quadrilaterals are squares?
6. What are the areas of these two squares in terms of a and b ? Calculate these two areas.
7. Explain why the diagrams show that $a^2 + b^2 = c^2$. Do your calculations agree with this?
8. What have you discovered about the areas of squares built on the sides of your right angled triangle?
9. Does this always work for all values of a and b ? Why?

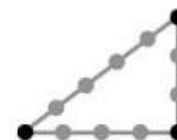


Notes

Ideally learners will have met the proof of Pythagoras Theorem given in Activity 2 in lower secondary school but it is included here for use where this is not the case or for review purposes.

Activity 3 Making right angles. The Converse of Pythagoras Theorem

Resources needed: Rope or string 4 metres long knotted at intervals of 10 centimetres. Tie knots in your rope before the lesson. It will be useful for other work on measurement.



- a. Get three learners to hold the rope to make a triangle with sides of length 3 units, 4 units and 5 units. Ask the class what they notice. Everyone should check that $3^2 + 4^2 = 5^2$ so that this triangle must be a right angled triangle. Repeat the same activity for a triangle with sides of length 6 units, 8 units and 10 units. Ask the learners what connection this triangle has with the 3-4-5 triangle. Repeat the same activity for a triangle with sides of length 5 units, 12 units and 13 units.
- b. Discuss with the class the difference between saying "If the triangle is right angled then $a^2 + b^2 = c^2$ " and "If $a^2 + b^2 = c^2$ then the triangle is right angled". Ask the class "What do you know in each case and what follows?" Get the class to make up some "If... then..." sentences of their own where "If P then Q" is true but "If Q then P" is false.

Classroom Activities for Learners

Activity 4 Proof of Pythagoras Theorem by Similarity

Time: 1 hour

What the teacher does

Draws the diagram on the board before the lesson.

Does **not** tell the learners that this is a lesson on Pythagoras Theorem.

Goes around the classroom checking the drawings that learners are making and helping them to make the drawing correctly.

Asks learners to identify the similar triangles in the diagram. Gives time for learners to discuss this in pairs.

Checks that learners have identified 3 triangles all similar to each other.

Asks: "Can you use similarity to give d in terms of b and c ?" Goes around the class checking that learners understand and can succeed.

Asks: "Can you use similarity to give e in terms of a and c ?" Goes around the class checking that learners can succeed.

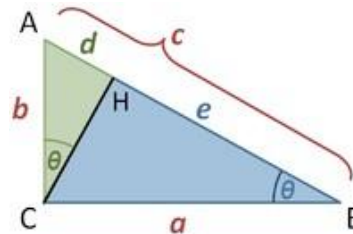
Asks "What do d and e add up to? What conclusion can you make?"

Gives time for learners to work independently, then to discuss in pairs. Facilitates a class discussion. Expects learners to make the connection with Pythagoras Theorem.

What the learners do

Angle ACB is a right angle.

CH is perpendicular to AB .



Learners copy the diagram carefully in pencil, erasing and making corrections if necessary.

Work independently, and then in pairs, to identify similar triangles. Record their results.

Work independently to find $d = b^2/c$ and then to find $e = a^2/c$. Record their results.

Discuss in pairs what they get from $d + e$ and the significance of this result. Record their results.

Join in a class discussion. Record their conclusions.

Ideas for teaching:

Don't rush the process of finding the proofs in these lessons. Don't tell your learners what to do. Give time for the learners to work out the proofs for themselves or in discussion with a partner. Good teaching results in learners developing problem solving and communication skills and independence. If your learners are dependent on you, and they expect you to tell them what to do, then ask yourself if you need to change your style of teaching.

The proof of the Cosine Formula as in Activity 3 in the Teacher Workshop can be introduced in a similar way as a separate lesson, giving the learners the diagram, asking them leading questions and allowing them to find the proof for themselves. The teacher should ask the class to discuss the connection between the Cosine Formula and Pythagoras Theorem. At this stage the proof of the Converse of Pythagoras Theorem can be introduced.

Make sure that all the learners understand the proofs of these results then give them some problems for practice in applying the results. If you have access to a computer then use the many good resources available such as Matter of Scale rich.maths.org/811 and the resources at www.cut-the-knot.org/pythagoras/

Discussion of Teaching Strategy – Developing Mathematical Reasoning and Proof

Rather than starting the lesson with a statement or proof of the theorem teachers should guide learners by asking questions about the diagrams and information given, encouraging learners to be active and curious, to spot patterns, to create their own conjectures, to ask their own questions, to experiment with ideas, to help one another, to set goals for themselves, to monitor their own progress and to know that making mistakes and ‘being stuck’ are part of learning. Later in the lesson teachers should base their explanations of the reasoning behind the proofs of the theorem on the learners’ own experiences.

Games can be used to encourage new ways of thinking and to engage learners in exploring ideas and pathways of reasoning new to them. According to (Scarfe 1962) “All play is associated with intense thought activity and rapid intellectual growth. The highest form of research is essentially play. Einstein is quoted as saying: the desire to arrive finally at logically connected concepts is the emotional basis of a vague play with basic ideas. This combinatory or associative play seems to be the essential feature in productive thought.”

As a lesson starter, the Squares Game can lead to Pythagoras Theorem or coordinates. It reinforces recognition of perpendicular directions and you can make the connection with ideas of gradient (distance up divided by distance across). As a class game it gives teachers the opportunity to act as judge when a square is claimed in error, or when a player fails to recognise a square, and to explain how to recognise a square by looking at corresponding distances across and up, the lengths of edges, the right angles and the symmetry.

The activities in this chapter provide scaffolding for learners to develop different proofs of Pythagoras Theorem. In addition the activities bring in the important ideas of generalizing a theorem, in this case the proof of the cosine rule as a generalization of Pythagoras Theorem, and also of proving both a theorem and its converse. To help learners to appreciate the need to prove the converse, teachers should introduce some statements of the form ‘If p then q ’ where the converse ‘if q then p ’ is false and the class should discuss these logical forms and suggest some examples of their own. Examples like the following are easy to invent:

- i) ‘if it is a fish then it is not a bird’ (true) and ‘if it is not a bird then it is a fish’ (false)
- ii) ‘if $x = 7$ then $x > 5$ ’ (true) and its converse: ‘if $x > 5$ then $x = 7$ ’ (false)
- iii) ‘if he is my brother then I am his sister’ (true) and if I am his sister then he is my brother (also true).

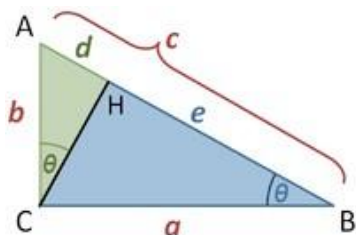
By playing with four copies of a right angled triangle, and rearranging them in different configurations as in Classroom Activity 2, learners can explore ideas of what stays the same and what changes between the two arrangements and discover for themselves facts about the areas of the squares on the sides of right angled triangles. As the same line of reasoning applies to any right angled triangle the learners will have discovered a proof of Pythagoras Theorem for themselves.

Learners can also be guided from step to step by questioning about similar triangles (as in the Key Questions below) to prove the theorem using similar triangles and then to generalise the proof to proving the cosine formula.

1962 November, Childhood Education, Volume 39, Issue 3, “Play is Education” by N. V. Scarfe, Start Page 117, Quote Page 120, Published Association for Childhood Education International, Washington D.C. (Verified with scans; thanks to Jacksonville, Florida public library)

Key Questions

Can you answer these questions? Can you give explanations for all your answers?



1. In this triangle angle ACB is a right angle and H is the foot of the perpendicular from C to AB.

Which triangles are right angled?

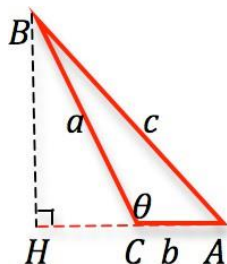
How would you prove that $\angle ACH = \angle ABC$?

Which triangles are similar and why?

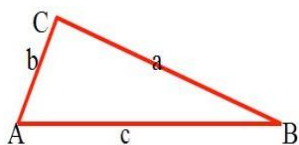
Explain the reasons that you can write d in terms of b and c.

Explain the reasons that you can write e in terms of a and c.

Explain how these results lead to a proof of Pythagoras Theorem.



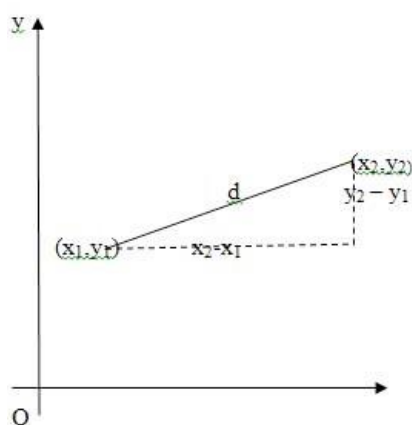
2. How would you change the proof of the Cosine Formula for the case where $\angle ACB$ is obtuse?



3. In triangle ABC, what can you say about $\angle ACB$ in the three cases:

- i. $c^2 < a^2 + b^2$
- ii. $c^2 = a^2 + b^2$
- iii. $c^2 > a^2 + b^2$

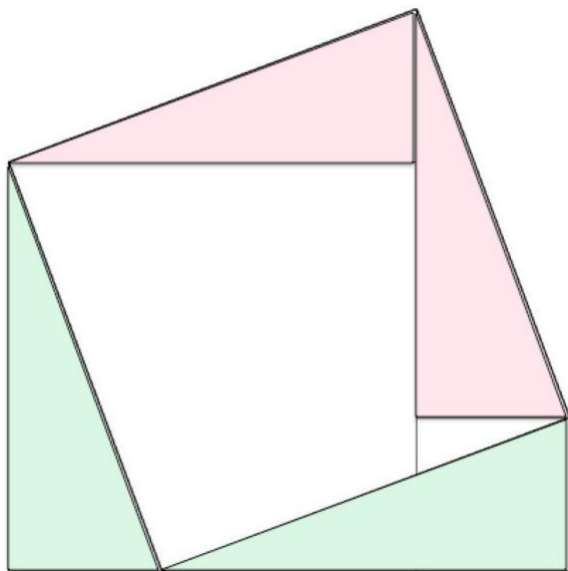
4. Make up 3 statements where the statement is true and the converse is false.



5. Explain how the formula for the distance between two points in the Cartesian plane is simply another way of stating Pythagoras Theorem.

Make Posters

Give different groups of learners the task of making posters to explain different proofs of Pythagoras Theorem. The learners can then be given the task to explain their proof to the class, either using their poster or using a powerpoint presentation. The posters can then be put up on the classroom wall to help learners to remember the reasoning behind the proofs.



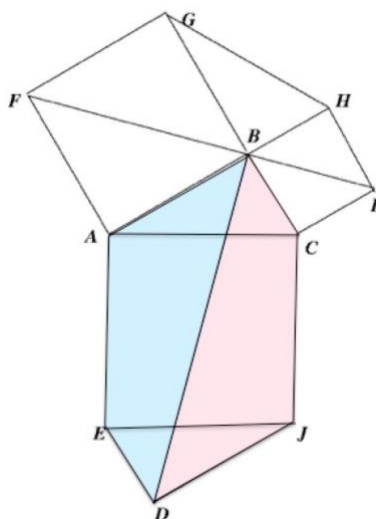
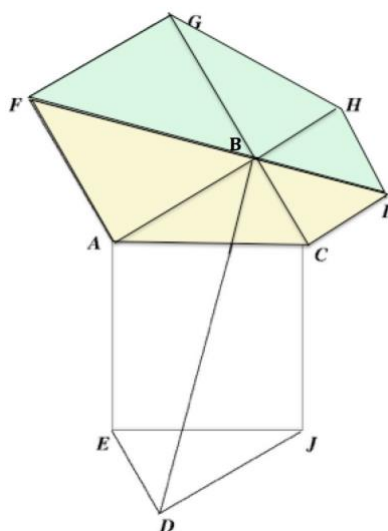
Perigal's Dissection

This diagram provides a proof without words of Pythagoras Theorem.

Can you see it?

Remove first the two green squares. What do you see?

Replace the green triangles and remove the pink triangles. What do you see?



Leonardo da Vinci's Proof

The diagrams are drawn with squares on the sides of the right angled triangle ABC. Can you spot the two congruent copies of $\triangle ABC$?

Can you prove that the four quadrilaterals FGHI, FICA, ABDE and BCJD are all congruent and therefore equal in area?

Now think about the coloured hexagons that are each made up of two congruent quadrilaterals. They have equal areas.

Remove two copies of the right-angled triangle from the hexagon FGHICA and you are left with the squares on the two shorter edges of the right-angled triangle.

Remove two copies of the right-angled triangle from the hexagon ABCJDE and you are left with the square on the hypotenuse of the right-angled triangle.