AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE
TEACHER NETWORK http://aiminghigh.aimssec.ac.za

IDEAS FOR TEACHING MATHEMATICS WITH MULTILINK CUBES

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African Institute for Mathematical Sciences
Schools Enrichment Centre

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CONTENTS

21st century teaching method

STUDENT ACTIVITY
Mathematical Thinking
Using what they know
Trying out new ideas
Conjecturing and testing

↓

STUDENTS TALK

↓

STUDENTS RECORD

↓

CLASS DISCUSSION
TEACHER EXPLANATION

Teacher guides learning

ENQUIRY BASED LEARNING
OR
GUIDED RE-INVENTION

SORTING AND COUNTING

Learners can do all the following activities using the cubes, and many more activities.

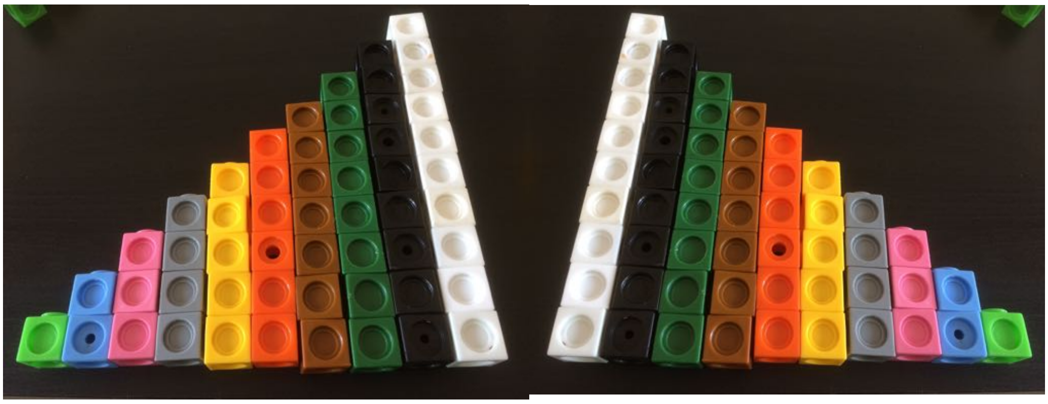
Teachers can ask these questions, and many more questions.

Using this concrete material helps young learners to understand mathematical concepts and to remember number facts.

The most effective teaching is learner centred **ENQUIRY BASED LEARNING**. Learners engage in exploring new ideas, building on what they already know, and thinking for themselves, guided by questions asked by the teacher.

1. Sort the cubes into colours and name the colours.
2. Count the number of cubes of each colour.
3. Arrange the piles in order of size from smallest to biggest.
4. Which is the BIGGEST pile? How many?
5. Which is the SMALLEST pile? How many?
6. Are there MORE white cubes than yellow cubes?
7. How many more?
8. Are there FEWER (or LESS) grey cubes than green cubes?
9. How many less?
10. Is the number of pink cubes BIGGER OR SMALLER than the number of white cubes?

MAKING NUMBER STICKS



Make sticks of lengths, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10.

1. Arrange the sticks in order.
2. Arrange in INCREASING order and DECREASING order.

NUMBER BONDS

Teachers can ask these questions, and many more:

Show me a 4-stick (a stick of length 4)

What goes with this stick to make 10?

What do the pink and brown stick make when you put them together?

Show me two sticks that add up to 10.

How many different pairs of numbers make 10?

What number do you get if you take 2 away from 10?

Learners build their number knowledge and understanding gradually using the concrete materials.

1 + 9 = 10	**10 – 1 = 9**	**10 – 9 = 1**
2 + 8 = 10	**10 – 2 = 8**	**10 – 8 = 2**
3 + 7 = 10	**10 – 3 = 7**	**10 – 7 = 3**
4 + 6 = 10	**10 – 4 = 6**	**10 – 6 = 4**
5 + 5 = 10	**10 – 5 = 5**	**10 – 5 = 5**
6 + 4 = 10	**10 – 6 = 4**	**10 – 4 = 6**
7 + 3 = 10	**10 – 7 = 3**	**10 – 3 = 7**
8 + 2 = 10	**10 – 8 = 2**	**10 – 2 = 8**
9 + 1 = 10	**10 – 9 = 1**	**10 – 1 = 9**
0 + 10 = 10	**10 – 0 = 10**	

ADDITION

9 + 1 = 10

1 + 9 = 10

9 + 1 = 10

1 + 9 = 9 + 1

You can also use this to show 10 – 1 = 9 and 10 – 9 = 1
but here we concentrate on addition.

8 + 2 = 10

2 + 8 = 10

8 + 2 = 10

8 + 2 = 2 + 8

ADDITION

$7 + 3 = 10$

$3 + 7 = 10$

$7 + 3 = 10$

$7 + 3 = 3 + 7$

$6 + 4 = 10$

$4 + 6 = 10$

$6 + 4 = 10$

$6 + 4 = 4 + 6$

$5 + 5 = 10$

SUBTRACTION

10 – 1 = 9

10 – 9 = 1

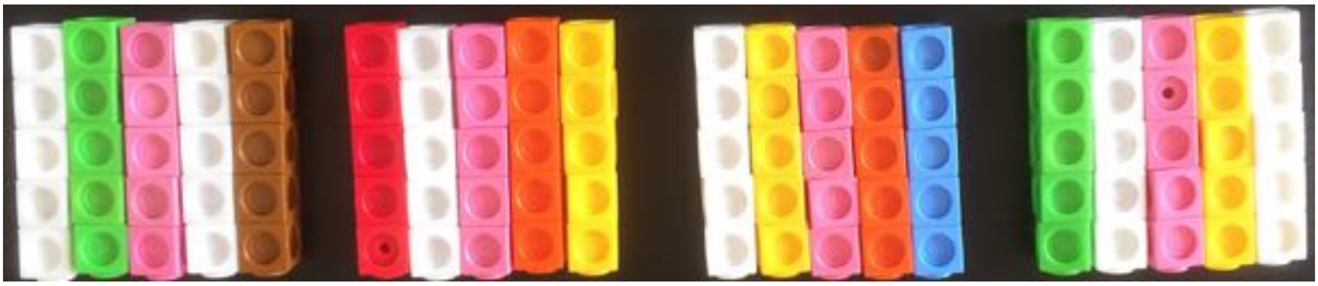
10 – 2 = 8

These are just a few examples. There are endless possibilities for using the cubes to show other number relations.

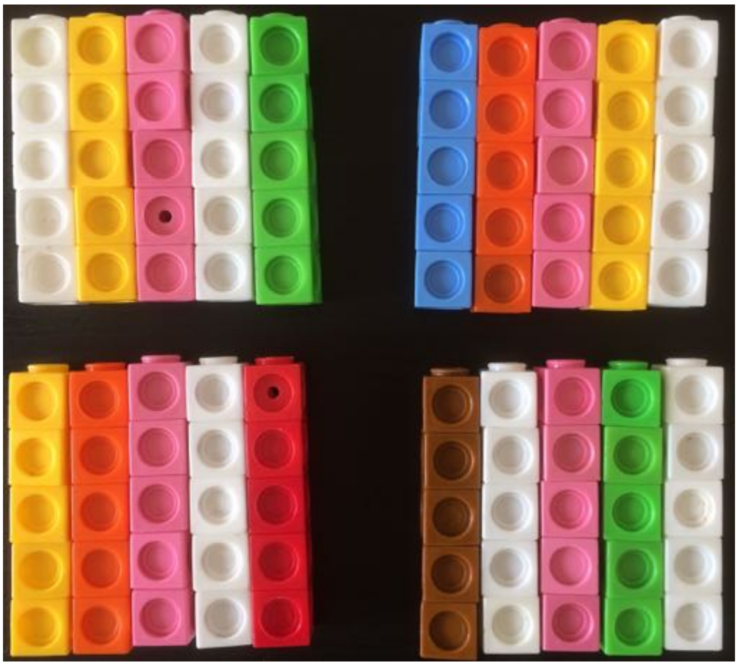
Teachers can show the number sticks and number relations and ask learners **to say** what the number sticks are showing and **to write down** what they show. In this way learners are actively developing their knowledge of the mathematical language and symbols and building their understanding of the numerical operations.

COUNTING IN 5s – MULTIPLES

5– 10 – 15– 20 – 25– 30 – 35– 40 – 45– 50 – 55 – 60 – 65 – 70 – 75 – 80 – 85– 90 – 95 - 100



$$20 \times 5 = 5 \times 20 = 100$$



$$5 \times 5 = 25$$

$$4 \times 25 = 100$$

$$10 \times 10 = 100$$

This is just one example of counting to give multiples and multiplication tables. Use similar methods for counting in 2s, 3s, 4s, 6s etc.
Also show square numbers 1, 4, 9, 16, 25, …

MULTIPLICATION

$$5 \times 2 = 10$$

$$3 \times 3 = 9$$

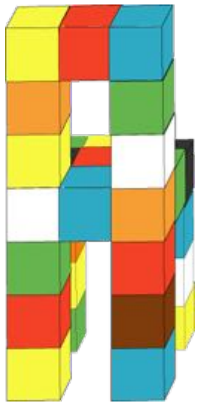
$$2 \times 4 = 8$$

CHAIR AND TABLE

I made a chair from cubes that looks like this.

Can you make a chair too? It need not be the same as mine.

Now can you make a table to go with your chair?



Then make chairs for the Three Bears:

a big chair, a middle-sized one and small one.

SOLUTION

Here are some chairs and tables.

Many different chairs and tables are possible.

And here you see a big chair, a medium sized chair and a small chair

NOTES FOR TEACHERS

Why do this activity?

Although this activity seems straightforward, learners will find scaling the table to match the chair quite a challenge. Tackling this will help to develop learner's spatial awareness. It is also a great opportunity to encourage discussion among the class or group.

Intended learning outcomes

PRIMARY

Development of special awareness.

SECONDARY

Understanding of scale of enlargement in 3 dimensions.

Suggestions for Teaching

A hands-on approach is absolutely necessary for this problem. You could start by making a chair from interlocking cubes yourself and showing the group, and then challenging them to make one of their own. (Do not make a chair that looks too good - some children find this discouraging!) You might like to put your chair somewhere so it can be seen easily and learners can refer to it as they build their own. Alternatively, you may decide to hide it at first to see how learners get on.

Learners could then work in pairs or small groups to make a chair and a table under which the chair will fit. This should promote much discussion about space and shape. At the end the whole group could come together to see each other's handiwork.

Key questions

How many cubes have you used for that leg of the chair?
How many cubes will you use for the seat of the chair?
Does your chair fit under the table with enough space for someone to sit down?
Are your 3 chairs similar? Are the 3 chairs in the same proportions?
Are the 3 chairs enlargements?
How many cubes in each chair?

Possible extension

- Make three chairs and three tables in different sizes
- Draw the 3 chairs
- Make 3 chairs that are in **the same proportions** but **enlargements of the smallest chair**. For example the chairs could have
    - legs 4 blocks high and a 4 by 4 seat
    - legs 3 blocks high and a 3 by 3 seat
    - legs 2 blocks high and a 2 by 2 seat

Possible support

You could encourage those who are struggling by asking them to make the legs first and then a seat for them fit into. Finally, they could put a back onto this "stool".

FACTORS AND MULTIPLES

$5 \times 2 = 10$	$3 \times 2 = 2 \times 3 = 6$
So 10 is a **multiple** of 2 and 5 2 and 5 are **factors** of 10	6 is a **multiple** of 2 and of 3 2 and 3 are **factors** of 6

DIVISION

How many 2's are there in 10? **$10 \div 2 = 5$**
Share 10 objects between 5 people, how many do they each get? **$10 \div 5 = 2$**
Divide 10 into 5 equal parts – how many in each part? **$10 \div 5 = 2$**

How many 3's are there in 9? **$? \times 3 = 9$**
Share 9 objects between 3 people, how many do they each get? **$9 \div 3 = 3$**
Divide 9 into 3 equal parts – how many in each part? **$9 \div 3 = 3$**

How many 3's are there in 6? **$? \times 3 = 6$**
Share 6 objects between 3 people, how many do they each get? **$6 \div 3 = 2$**
Divide 6 into 3 equal parts – how many in each part? **$6 \div 3 = 2$**

How many 2's are there in 6? **$? \times 2 = 6$**
Share 6 objects between 2 people, how many do they each get? **$6 \div 2 = 3$**
Divide 6 into 2 equal parts – how many in each part? **$6 \div 2 = 3$**

FRACTIONS

TWELFTHS $\frac{1}{12}$

SIXTHS $\frac{1}{6}$

QUARTERS $\frac{1}{4}$

THIRDS $\frac{1}{3}$

HALVES $\frac{1}{2}$

ONE UNIT

The fraction wall shows for example $\frac{1}{2} = \frac{3}{6} = \frac{6}{12}$,

$\frac{2}{3} = \frac{4}{6} = \frac{8}{12}$,

$\frac{1}{3} = \frac{2}{6} = \frac{4}{12}$.

This is 1 unit. The white cubes are sixths written: $\frac{1}{6}$

The picture shows:

2 halves make one unit $2\times\frac{1}{2} = 1$

One half is equivalent to three sixths $\frac{1}{2} = \frac{3}{6}$

3 thirds make one unit $3\times\frac{1}{3} = 1$

One third is equivalent to two sixths $\frac{1}{3} = \frac{2}{6}$

Adding a half plus a third we use the equivalent fractions

$$\mathbf{\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}}$$

ODD NUMBERS

$1 + 3 = 4$

$2 \times 2 = 4$

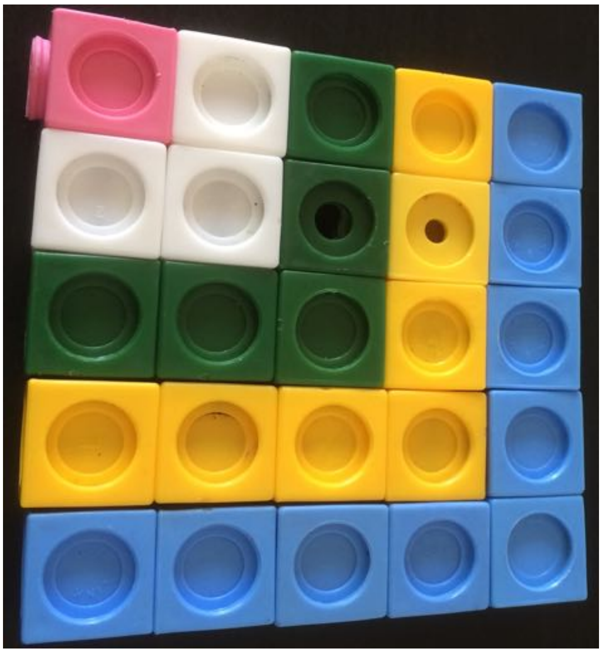
$1 + 3 + 5 = 9$

$3 \times 3 = 9$

$1 + 3 + 5 + 7 = 16$

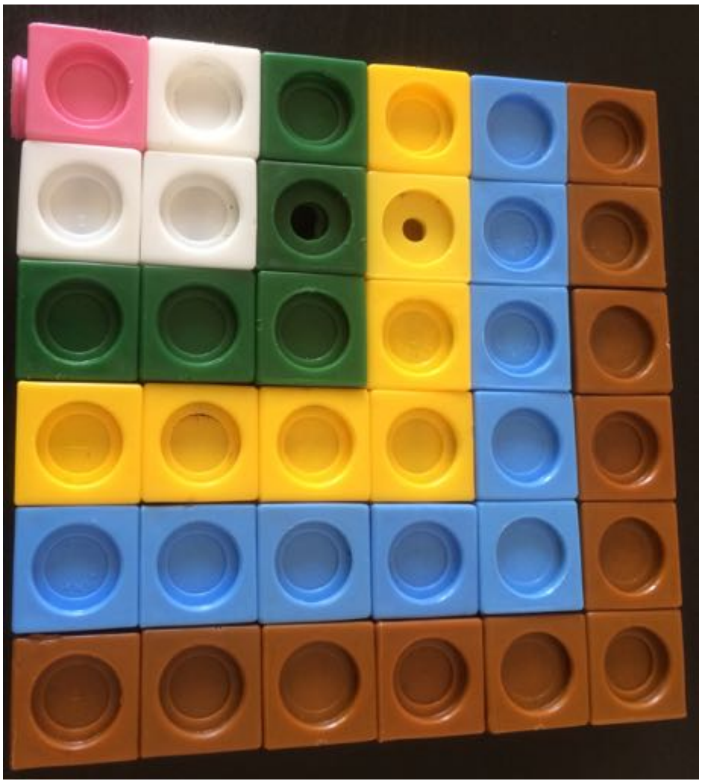
$4 \times 4 = 16$

Learners should be given the opportunity to explore number patterns and to find out for themselves what happens when they extend the patterns to larger numbers. Teachers should not tell the learners what they will find out but should ask questions like “What do you notice?”



$1 + 3 + 5 + 7 + 9 = 25$

$5 \times 5 = 25$



$1 + 3 + 5 + 7 + 9 + 11 = 36$

$6 \times 6 = 36$

ODD SQUARES

Think of a number and square it. What answer do you get?

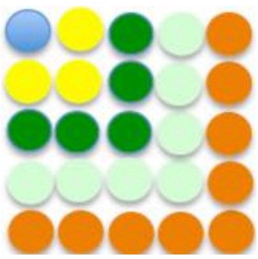
What do you notice about this picture?

Does the picture make you think of a number pattern?

Does the pattern work for all squares, 2 by 2, 3 by 3, …. and so on?

What would the pattern be for a 100 by 100 square?

SOLUTION



A picture similar to this can be drawn for any square array. The cubes or discs can be arranged in different colours, as they are shown in the picture, so that there is a number of each colour for all the odd numbers 1, 3, 5, … (2n-1).

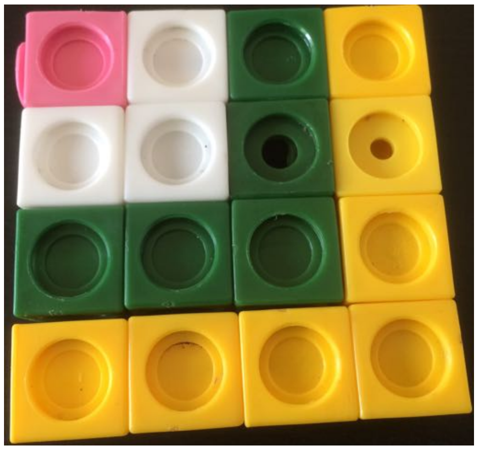
Adding up the odd numbers always gives a square number.
Algebraically this is: $1 + 3 + 5 + \ldots + (2n-1) = n^2$.

The pattern for a 100 by 100 square would be $1 + 2 + 3 + \ldots + 199 = 100^2 = 10\ 000$

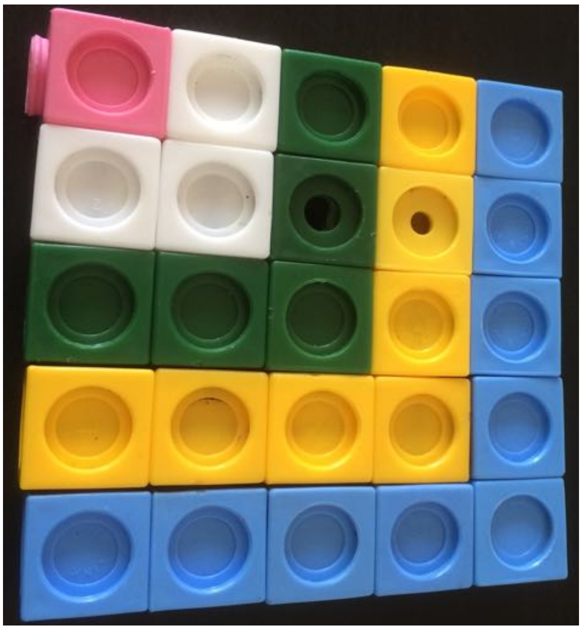
Young learners will be able to discover this for themselves. Learners working towards the school leaving examination leading to higher education should be able to prove the general result using algebra.

$1 + 3 = 4$
$2 \times 2 = 4$

$1 + 3 + 5 = 9$
$3 \times 3 = 9$

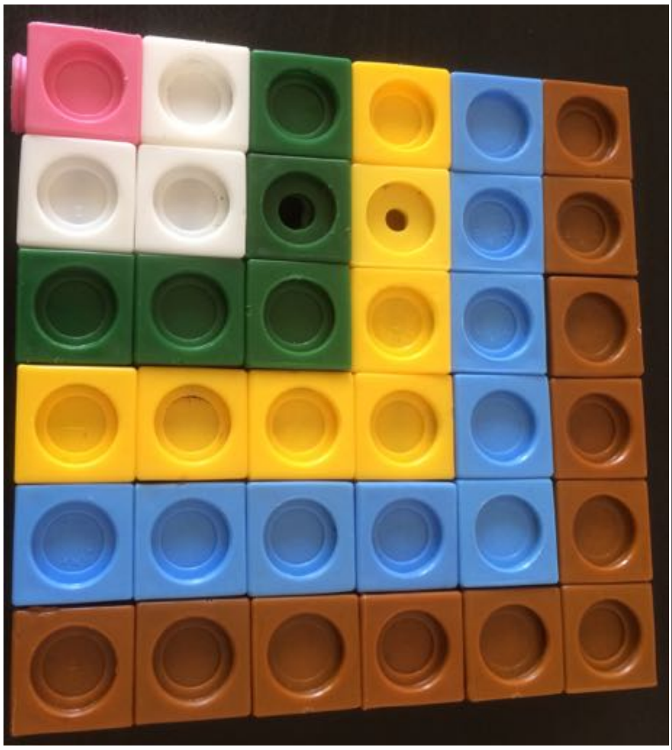


$1 + 3 + 5 + 7 = 16$
$4 \times 4 = 16$



$1 + 3 + 5 + 7 + 9 = 25$

$5 \times 5 = 25$



$1 + 3 + 5 + 7 + 9 + 11 = 36$

$6 \times 6 = 36$

Formal proof using Mathematical Induction

To Prove $1 + 3 + 5 + \ldots + (2n-1) = n^2$.

Proof

Suppose that the result is true for n = 1, 2, 3,… up to n, then

$1 + 3 + 5 + \ldots + (2n-1) = n^2$

and it follows that

$1 + 3 + 5 + \ldots + (2n-1) + (2n+1) = n^2 + (2n + 1)$

But we know $n^2 + 2n + 1 = (n + 1)^2$ so we have proved that

$1 + 3 + 5 + \ldots + (2n-1) + (2(n+1) - 1) = (n+1)^2$, which is the result for (n + 1).

So, by the Axiom of Mathematical Induction the result is true for all natural numbers.

NOTES FOR TEACHERS

Why do this activity?

This activity is a wonderful example of a context in which an informal proof is accessible to learners via an image with no algebra required. It would be a good choice to try with your learners once they are familiar with square numbers. Having concrete materials such as Multilink cubes enables learners to engage with the ideas in a 'hands on' way to help them to develop their own understanding of the mathematics involved.

As well as encouraging visualisation, it gives learners opportunities to conjecture, justify and generalise. This activity is suitable for learners of all abilities. All learners should have some success using words and pictures. Older and more able learners can then be encouraged to use algebra.

Intended learning outcomes

UPPER PRIMARY

- To investigate and extend numeric and geometric patterns looking for relationships between numbers, including patterns.

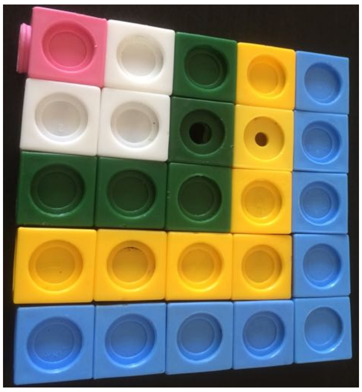
LOWER SECONDARY

- To investigate and extend numeric and geometric patterns looking for relationships between numbers, including patterns.
- To represent patterns algebraically.

UPPER SECONDARY

- To investigate and extend numeric and geometric patterns looking for relationships between numbers, including patterns.
- To represent patterns algebraically and to factorise and simplify algebraic expressions.
- To use logical reasoning and algebra to give a formal proof of the general result.

Suggestions for Teaching



You could introduce the activity by showing this pattern and asking the learners what they notice. Don't say "Yes, that's right" or "No …", just accept **all** responses. This encourages learners to make suggestions. Don't tell them that the numbers are odd or that the pattern shows a square number. After a few minutes ask the learners to work in pairs or small groups to make patterns like this and to write down what they notice about the patterns.

When the learners have had time to make the patterns and write down some results you could have a class discussion in which you ask learners to show their models and share their ideas. What do the class notice? Ask for comments on the arrangement of cubes of different colours. Ask "How can this help us explain the relationship between square numbers and the sum of odd numbers?"

Then give them time to discuss in pairs or small groups their answers to the following questions:
"How many more cubes will I need to add to make the next square? And the next? And the next?"
"How many more cubes will I need to go from the 99th square to the 100th?"
This is a chance for them to offer some suggestions, however 'polished' the explanation might be.

For a primary class it is sufficient that learners can find the pattern: 1
1+3
1+3 + 5
1+3 +5 + 7 and so on,
and that they can see that this sequence always gives a square.

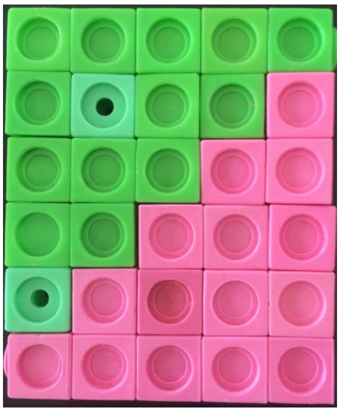
For a lower secondary class most learners should be able to find the answer for the 100 by 100 square while the most able learners should be able to find an algebraic expression for the general result, that is:
$1 + 3 + 5 + \ldots + (2n-1) = n^2$.

For a top class in secondary school you could ask learners to use algebra to prove the result.

It would be great to try and capture this for a display. You could jot down the steps of the explanations on the board as the learners build them up. Then the final versions could be put up on the wall with the problem and the images. It would be good to display any other proofs that the class has come up with.

Key questions
What do you notice about the result each time?
Will this always be the case? How do you know?
Can you describe what is happening in the picture?
Can you see any patterns?
Can you explain how the patterns work?
What is the smallest starting number you could start with in this pattern?
Can you draw similar pictures for bigger squares?

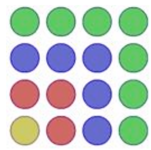
Possible extension



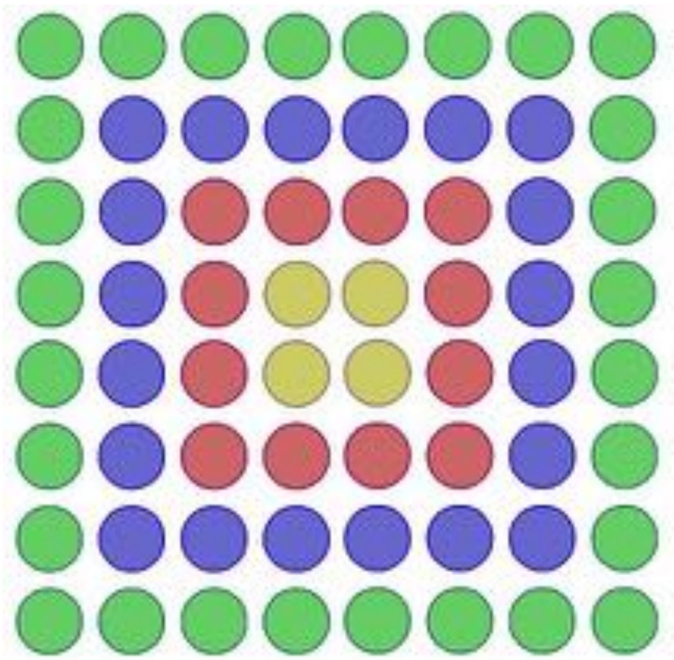
TRIANGLE NUMBER PICTURE
https://aiminghigh.aimssec.ac.za/grades-9-to-12-triangle-number-picture/
is another learning activity that focuses on visual proof for which Multilink models can be made. Although it leads into algebra, many learners will be able to offer explanations and informal proofs.

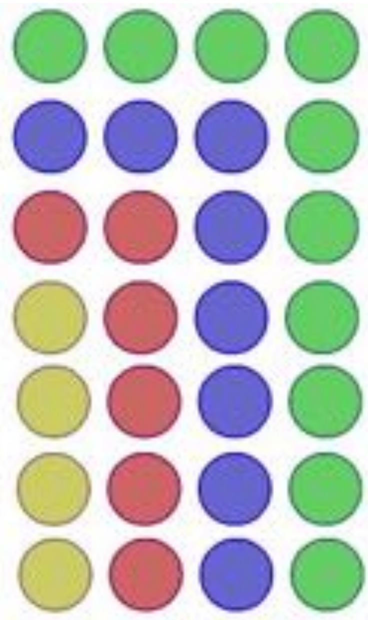
Possible support
Some learners at all stage might find it useful to use counters or cubes to represent the numbers and therefore to build up a picture of what is going on in this way. Also, when the learners work in pairs or small groups encourage the learners to explain their ideas to each other.

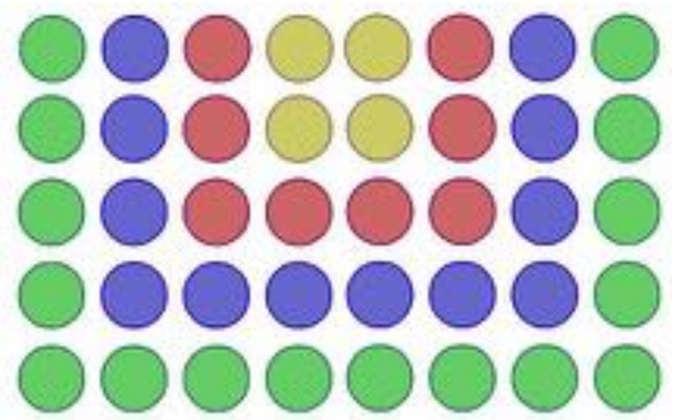
One excellent teaching strategy, at the point in a lesson when some learners have found a solution and others have not, is to ask a learner from a group who have found a solution and a good explanation to change places with a learner from a group that is struggling. Then in both groups there will be a learner or learners to explain to the others and to help them to understand.



These further challenges come from
www.nrich.maths.org

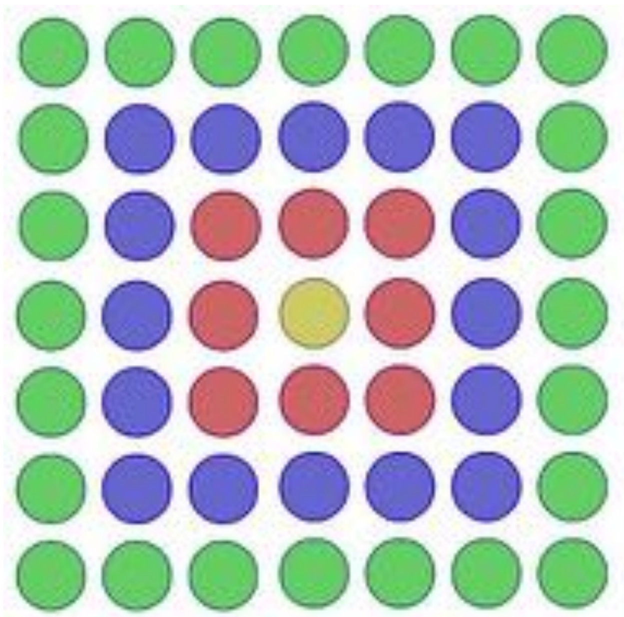






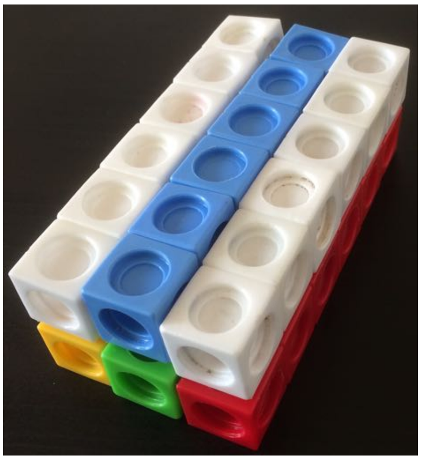
We have just seen that the sum of odd numbers can be represented as a square array of dots.
The square diagram can be adapted to represent related summations.
Design your own, or use some of ours, and make notes on:

- How it was built from the original square array.
- How many dots there would be in the 100^{th} diagram.
- What summation it represents.
- Whether there are alternative ways of expressing the summation, relating to different ways of looking at the diagram.

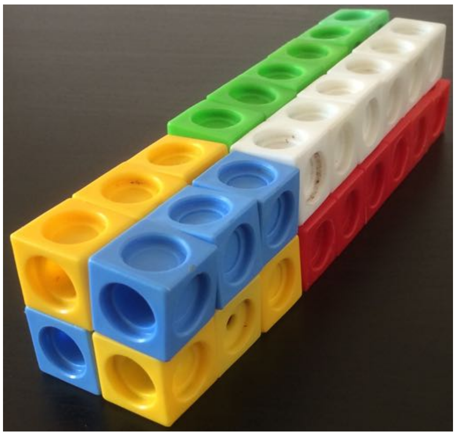


CUBES AND CUBOIDS

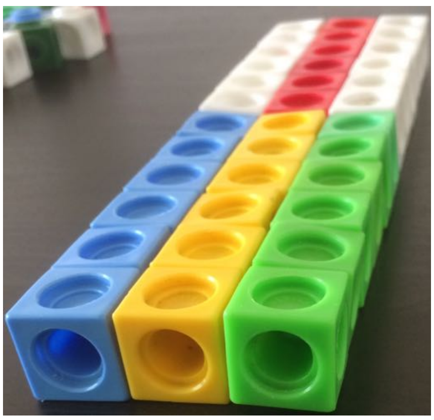
$5 \times 5 \times 5 = 125$



$6 \times 3 \times 2 = 36$



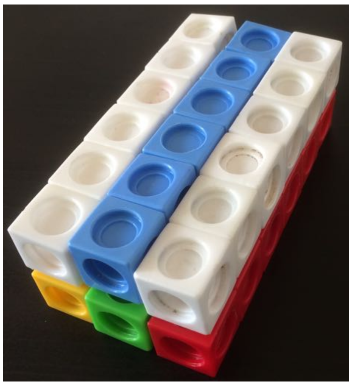
$9 \times 2 \times 2 = 36$



$12 \times 3 \times 1 = 36$

$18 \times 2 \times 1 = 36$

CUBOIDS



The picture shows a cuboid made from 36 cubes.

How many different cuboids can you make from 36 cubes?

Do they all have the same volume? Why or why not?

Do they all have the same surface area? Why or why not?

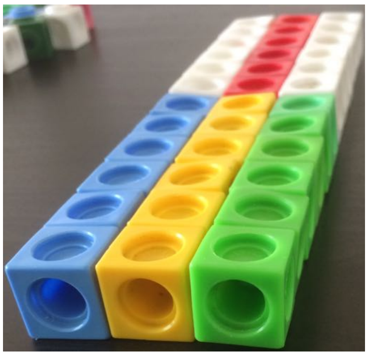
Take the area of each face of the cube as one square unit of area and find the surface area of all the cuboids that can be made using 36 small cubes. Which one has the smallest surface area?

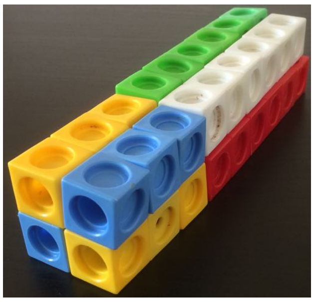
SOLUTION

The pictures show a few of the solutions.

The solutions are:

Cuboid	Volume in cubic units	Surface area in square units
36 × 1 × 1	36	146
18 × 2 × 1	36	112
12 × 3 × 1	36	102
9 × 4 × 1	36	98
9 × 2 × 2	36	80
6 × 6 × 1	36	96
6 × 3 × 2	36	72
4 × 3 × 3	36	66





The cuboid with smallest surface area is the 4 by 3 by 3 cuboid.

NOTES FOR TEACHERS

Why do this activity?

This is a good activity to help learners to develop an understanding of the properties of a cuboid and of surface area and volume. It also gives learners practice in working out all the possible factors of a given number.

Intended learning outcomes

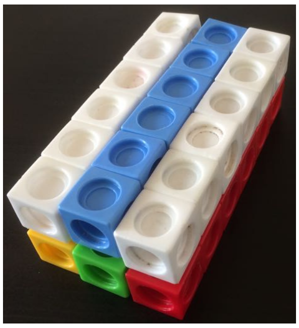
UPPER PRIMARY

- To develop the skills of visualisation and systematic working.
- To develop understanding of surface area and volume.

SECONDARY

- To develop the skills of visualisation and systematic working.
- To apply knowledge of factors of 36 and to work systematically to check that they have all possible triples of factors.
- To develop understanding of surface area and volume.

Suggestions for Teaching



You could start by showing the class one of the models and asking them to say how many small cubes have been used to make it. You could give them a few minutes to discuss this question in pairs.

When the class has decided that there are 36 cubes say that each cube has a volume of 1 cubic unit and the area of each face is 1 square unit. Ask the class to find the volume and surface area. Have a class discussion about finding the surface area BY COUNTING SQUARES until all the learners understand how to do this. Do not tell them to use a formula length × breadth × height – in fact it is not necessary to use this formula.

The class could make a poster to show all the solutions. If they do not find them in the first lesson you could suggest that they keep searching. Whenever a learner finds a new solution congratulate him or her warmly and add the solution to the poster.

Making a list or table is an important and useful way of checking that all possibilities have been included. To round off this activity you could show the group how to do this methodically so that none are included twice, such as $9 \times 4 \times 1$ and $4 \times 9 \times 1$. One way to do this is to list in order of the size of the numbers so that the above example will always be recorded as $9 \times 4 \times 1$.

Key questions

Can you split the 36 cubes into shorter equal lengths?
How many cubes are there in the top layer?
How many layers? How many small cubes all together.
How many small squares on that face?
Can you find the areas of all 6 faces?

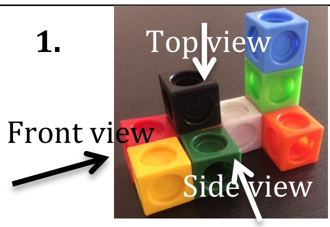
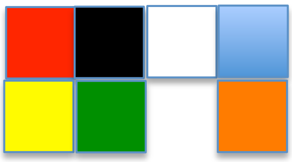
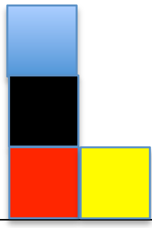
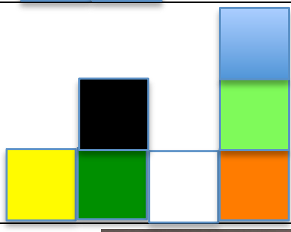
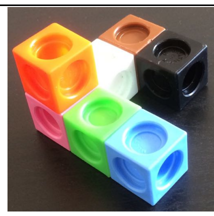
Possible extension

Can you find a cuboid (with edges of integer values) that has a surface area of exactly 100 square units. Is there more than one? See Cuboids https://aiminghigh.aimssec.ac.za/grade-9-or-10-cuboids/

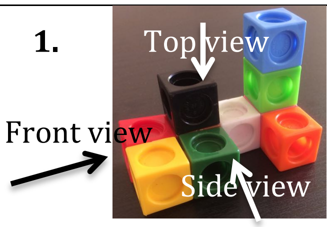
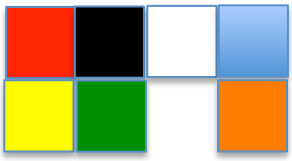
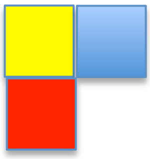
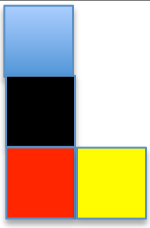
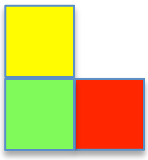
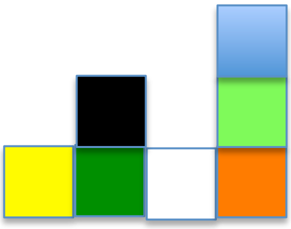
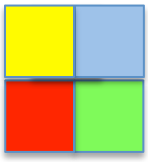
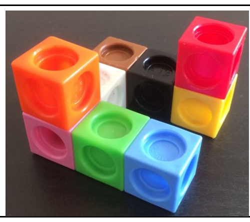
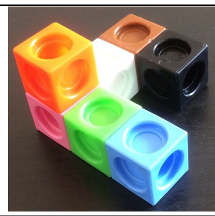
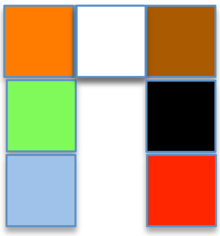
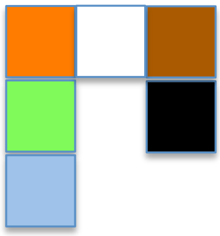
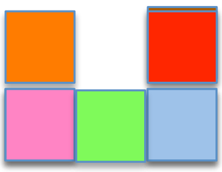
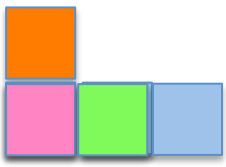
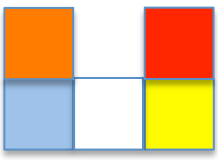
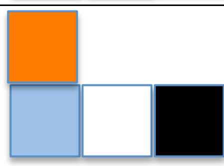
Possible support

Suggest that learners who are doing well help learners who are having difficulties.

TOP SIDE AND FRONT VIEWS

Make these objects. Draw the top view, side view and front view of each object. The first one has been drawn for you.	**1.** 	**2.** 
TOP VIEW	 **1. Top view**	**2.**
FRONT VIEW	 **1. Front view**	**2.**
SIDE VIEW	 **1. Side view**	**2.**
	3. 	**4.** 
TOP VIEW	**3.**	**4.**
FRONT VIEW	**3.**	**4.**
SIDE VIEW	**3.**	**4.**

SOLUTION TO TOP SIDE AND FRONT VIEWS

Make these objects. Draw the top view, side view and front view of each object. The first one has been drawn for you.	**1.** 	**2.** 
TOP VIEW	 **1. Top view**	**2.** 
FRONT VIEW	 **1. Front view**	**2.** 
SIDE VIEW	 **1. Side view**	**2.** 
	3. 	**4.** 
TOP VIEW	**3.** 	**4.** 
FRONT VIEW	**3.** 	**4.** 
SIDE VIEW	**3.** 	**4.** 

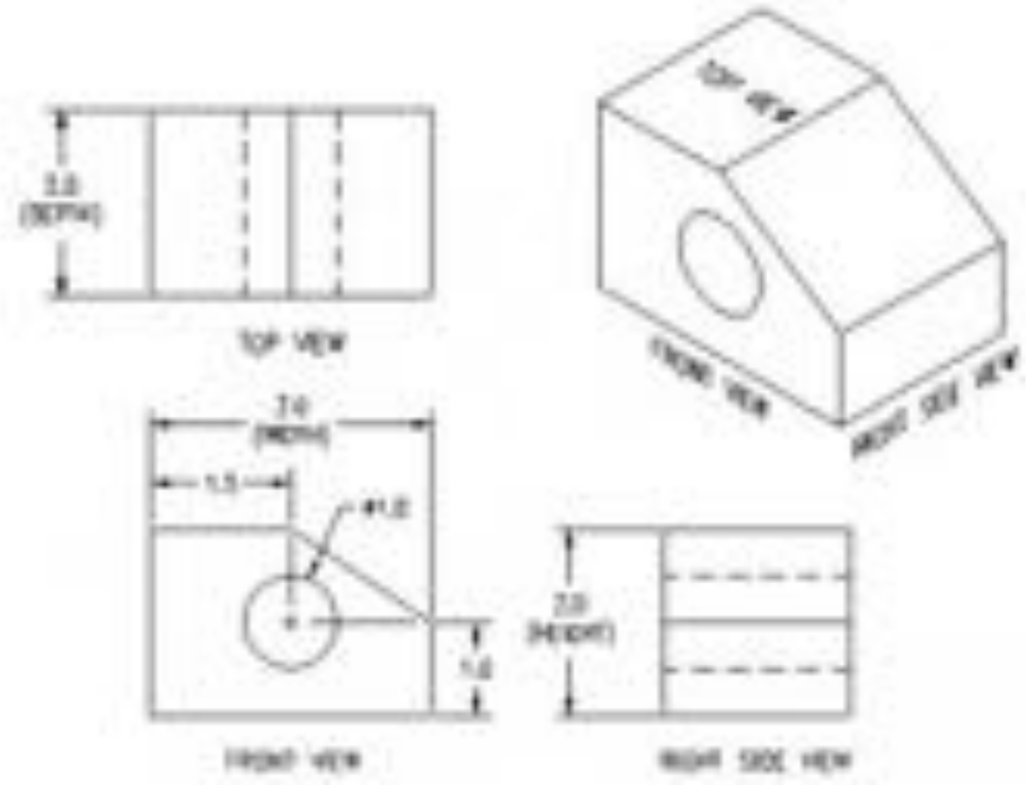
NOTES FOR TEACHERS

Why do this activity?

The skill of drawing and interpreting different views of the same object is essential to many human activities. This simple activity teaches learners the basic principals of plan and elevation drawings and provides learners with the opportunity to develop these important visualisation skills and to draw and interpret diagrams. The 'real world' applications include:

- building plans and architectural diagrams
- engineering drawings and accurate plans
- interpreting body scans and other images of the human body in medicine and surgery
- interpreting assembly diagrams for putting together flat packs of furniture or other manufactured items such as a bicycle that arrives in parts
- making toy models from instructions (e.g. Lego, Meccano)

At some stage older learners could be shown and discuss this example of an engineering drawing:



Intended learning outcomes

UPPER PRIMARY

- To understand the basic principals behind top, side and front view drawings and to be introduced to the language used by designers, architects and builders of ***plan and elevation drawing***.
- To develop visualisation and observation skills.
- To develop skills in drawing and interpreting diagrams.

SECONDARY

- To understand the basic principals behind plan and elevation drawing with more complicated objects.
- To develop visualisation and observation skills.
- To develop skills in drawing and interpreting diagrams.

Suggestions for Teaching

Cubes such as Multilink. isometric paper and squared paper would be useful but not essential.

You might like to make copies of page 1 for worksheets. If you have cubes available you might introduce the activity by showing the learners a model and turning it around to show the three views, then sketching them on the board.

This is a suitable activity for small groups of learners. It is helpful for them to have cubes to construct the models.

Key questions

What can you see from that direction?
What is hidden when you look at the object from that direction?

Possible support

It may help to look at some familiar objects from different directions and talk about what you see from those directions, for example a shoe and a mug.




Possible extension

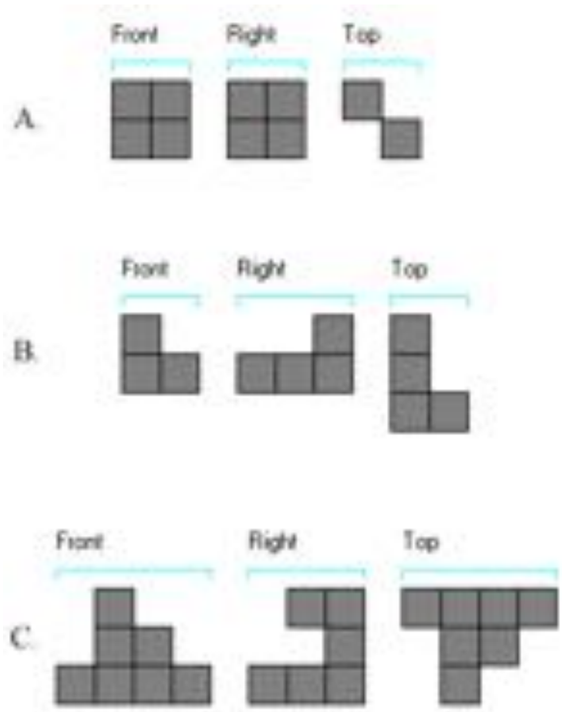

Figure A


Figure 1


Figure 2


Figure 3

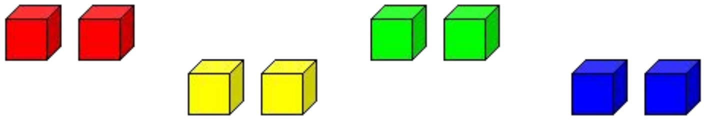

Figure 4



Here are more isometric views of different models. Learners can make the models from multilink and then draw plan and elevation drawings of the three views.	Here are some more plan and elevation drawings/ Learners can try to make the models and then to draw sketches of them.

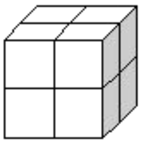
Also see the THREE VIEWS activity on the AIMINH HIGH website
https://aiminghigh.aimssec.ac.za/grades-4-to-8-three-views/

FOUR COLOURS



Akiki has 8 cubes, 2 red, 2 yellow, 2 green and 2 blue.

He wants to fit them together to make a cube so that each colour shows on each face just once.



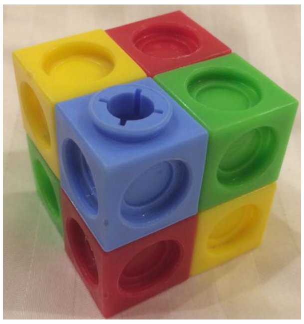
Is it possible ?

If so, how?

If you've found a way, is there another way?

SOLUTION

4 colours



	BACK FACE R Y B G	
	TOP FACE Y R B G	BOTTOM G B R Y
LEFT FACE Y B G R	FRONT FACE B G R Y	RIGHT FACE G R Y B

9 colours

Top layer
R Bl Y
O P W
B Br G

Middle layer
W O P
G B Br
Y R Bl

Bottom layer
Br G B
Bl Y R
P W O

NOTES FOR TEACHERS

Why do this activity?

Learners enjoy solving this puzzle. It introduces them to some logical reasoning as well as being solvable by trial and improvement.

Intended learning outcomes

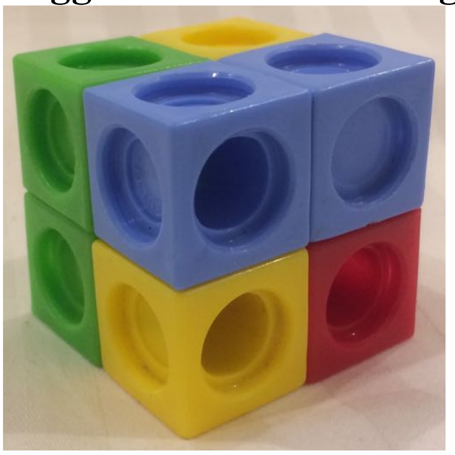
UPPER PRIMARY

- Development of spatial awareness

SECONDARY

- Development of spatial awareness
- Extension to 3 by 3 by 3 cube
- Development of logical thinking and systematic working

Suggestions for Teaching



You could put eight cubes (of the correct colours) together to make a large cube without there being one of each colour on each face. Ask learners what they notice. Give time for them to consider this individually then suggest they talk with a partner. Finally, you can draw the whole group together to share ideas.

Hopefully the initial discussion will have brought up some of the attributes of the large cube so you can introduce learners to the challenge itself. Plenty of cubes will be required for learners to try out their ideas.

After a short time, bring everyone together for a 'mini plenary' where progress is discussed. This can help learners articulate their ideas so far and it gives others a helping hand if they have found it hard to get started.

In the plenary, you could ask a few learners to share some particularly useful ways of approaching this problem, perhaps because they have worked systematically.

Key questions

How are you trying to solve this?
Have you checked each face has one of each colour cube?

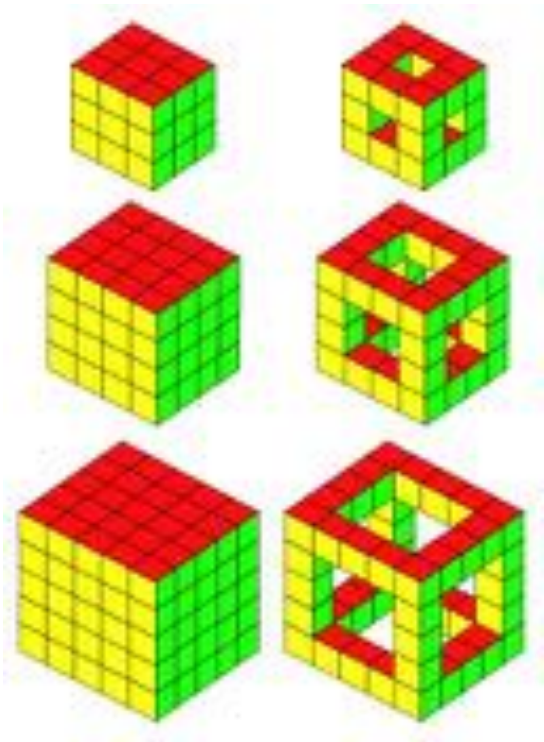
Possible support

Learners will find it easier to solve this puzzle if the can make the cube with multilink.

Possible extension

Try the puzzle for a 3 by 3 by 3 cube made from 27 smaller cubes with 9 colours and 3 of each colour.

HOLES IN CUBES



Here we have three solid cubes and three cubes that have holes.

They are just the first three in a series that could go on and on.

I was wondering about the number of cubes used in each ...

Then I thought about the difference between those numbers.

So, for example, I found that the first cube, 3 by 3 by 3, used 27 cubes.

The same cube with holes used 20 cubes.
Let’s call these types of cubes "Solid" and "Frame".

Explore the numbers of cubes for the next few Solid and Frame cubes?

Do you notice any patterns?

Can you explain any of the patterns?

SOLUTION

Length of edge	Number of cubes in solid	Number of cubes in frame Each time an extra cube is added to each of the 12 edges Each time there are 8 corner cubes	Difference A square of cubes is taken from 6 A cube of cubes is taken from the centre
3	3 x 3 x 3 = 27	12x1 + 8 = 20	7 = 6 x 1 + 1
4	4 x 4 x 4 = 64	12x2 + 8 = 32	32 = 6 x 4 + 8
5	5 x 5 x 5 = 125	12x3 + 8 = 44	81 = 6 x 9 + 27
6	6 x 6 x 6 = 216	12x4 + 8 = 56	160 = 6 x 16 + 64
7	7 x 7 x 7 = 343	12x5 + 8 = 68	275 = 6 x 25 + 125

NOTES FOR TEACHERS

Why do this activity?

This activity gives learners a chance to explore, to discover, to analyse and to communicate. It allows learners to use their own methods in whatever way they find most helpful. It also helps to develop visualising skills.

Intended learning outcomes
SECONDARY

- Development of visualising skills
- Recognition of number patterns
- Development of logical reasoning and ability to explain why the patterns occur
- Development of oral and written communication skills

Suggestions for Teaching

You could show the class models of the first solid cube and first frame cube. Invite them to talk to a partner about them and to share their observations with the whole class. You could then ask pairs to suggest what the 'next' two cubes would look like. You could also ask them to explain why they think you started with a 3 by 3 by 3 cube rather than, say, a 2 by 2 by 2 one.

You may like to challenge learners to find the number of small cubes each is made up of and they could share their methods for doing so. Some learners may need to make their own model to help them, but others will be able to calculate the number of cubes using what they see in the diagrams and facts they know. By sharing different methods, some children may take on a new method because they find it works better for them compared with the original way they chose. learners can then work on the challenge in pairs or small groups.

Let class the choose their own way of recording their findings and share these in the plenary, as well as sharing results. Encourage explanations of the number patterns, rather than just 'pattern spotting'.

Key questions

Tell me about what you've found.
Are there other ways of showing what you've found?
How many small cubes are taken from the faces of that one?
How many small cubes are taken from inside that one?

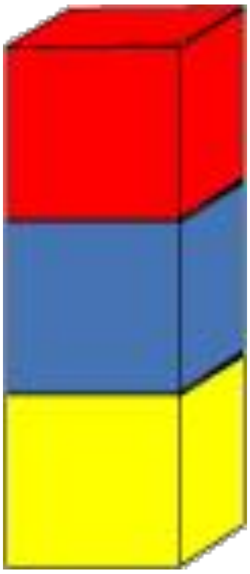
Possible extension

Learners could explore what happens when they count the square surfaces that are visible on each small cube and how many cubes are taken out of the inside of the solid cube.

Possible support

Have plenty of multilink cubes available to help learners if they need them.

THREE BLOCK TOWERS



Take three different colour blocks, maybe red, yellow and blue. Make a tower using one of each colour.

Now make a tower with a different colour on top.

How many **different** towers can you make?

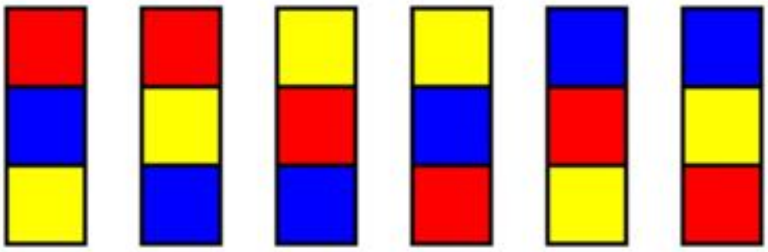
When you are sure you have found them all, try it with four colours.

What if you make towers with five colours.

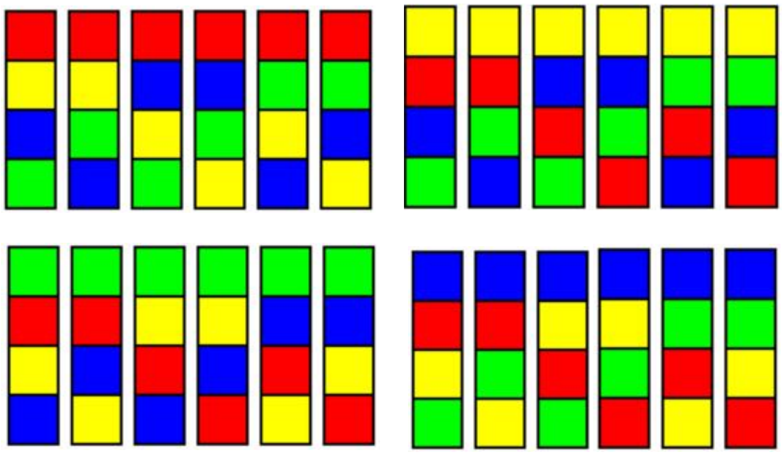
You might record your results like this:

R	B		or like this: RBY BYR …
B	Y	…	
Y	R		

SOLUTION TO THREE BLOCK TOWERS
LEADING TO PERMUTATIONS AND FACTORIALS



For 3 colours you can make 6 different towers.



For 4 colours you can make 24 different towers.

For 5 colours you can make 120 different towers.

For 6 colours you can make 720 different towers

For 3 colours you can have RBY RYB YRB YBR BRY BYR
3 choices for the 1st colour, 2 choices for the 2nd and 1 choice for the 3rd making $3 \times 2 \times 1 = 6$ different towers.

For 4 colours you have 4 choices for the 1st colour, 3 choices for the 2^{nd}, 2 choices for the 3rd colour and 1 choice for the 4^{th} colour
making $4 \times 3 \times 2 \times 1 = 24$ different towers.

For 5 colours you have 5 choices for the 1st colour, 4 choices for the 2^{nd}, 3 choices for the 3rd colour, 2 choices for the 4^{th} colour and 1 choice for the 5^{th} colour
making $5 \times 4 \times 3 \times 2 \times 1 = 120$ different towers.

For 10 colours you get $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 =$ 3, 628, 800 different towers.

We call these different ***arrangements*** or ***permutations*** of the colours.

The number of different arrangements of n different objects in a row is
$n(n-1)(n-2)(n-3) \ldots 3 \times 2 \times 1$ and we call this ***factorial n***, written **n!**

Can you find this on your calculator?

$2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$ $7! = 5040$ and so on

NOTES FOR TEACHERS

Why do this activity?

There are two parts to this activity, firstly, making as many different towers as you can think of, and secondly, making sure that you have made **all possible** towers (or arrangements). The first is easy but the second requires systematic working and logical thinking.

This activity can be adapted for learners of different ages.

Learning objectives

Lower Primary - age 6 to 8 Logical thinking and concrete experience to understand why there 6 different arrangements.

Upper Primary - age 9 to 11 Logical thinking and concrete experience

- to generalise the problem to 4 different colours,
- to make a clear record of all the arrangement,
- to understand why there 24 different arrangements
- and to give a clear explanation of the reasons.

Lower Secondary - age 12 to 15 As for the upper primary age group but more able learners should recognise the number pattern and be able to generalise the activity for larger numbers and to give the answer as an algebraic formula.

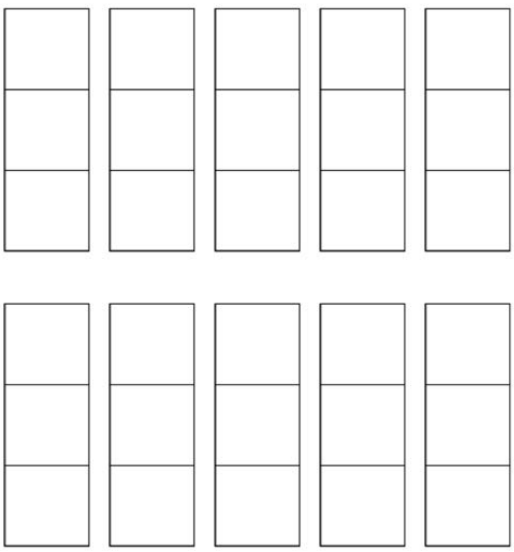
Upper Secondary - age 18 to 18

- To recognise the general situation and to understand that the number of arrangements or permutations of n different objects is **factorial n** written **n!**
- To be able to apply this to other problems including probability problems.

Suggestions for Teaching LOWER PRIMARY

For younger learners you could give 3 Multilink cubes of 3 different colours to each child (or pair of children) and ask them to find out how many different towers they can make.

You might like to give them squared paper or copies of this template so that they can record their findings.



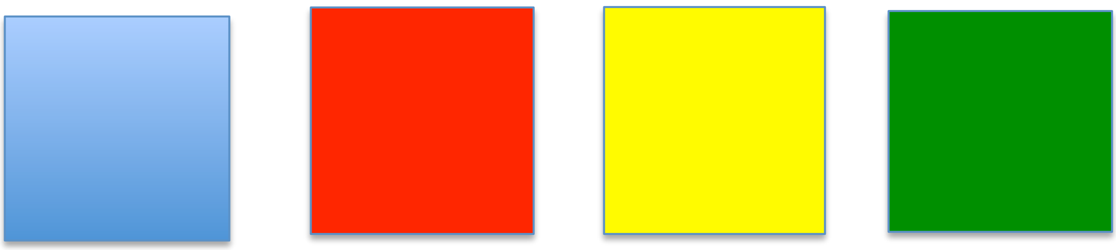
When the learners have had time to explore the different possibilities for themselves have a class discussion in which you invite learners to come to the front and make a three-block tower that's different from the tower made by the person before. Check in each case why it is different, encouraging the learners to do the explaining. Keep going until you have six different towers, and then ask another learner or two to come and make another and different tower. They will of course find it impossible, which gives you a nice way in to ask "Have we got them all? How can we be sure?".

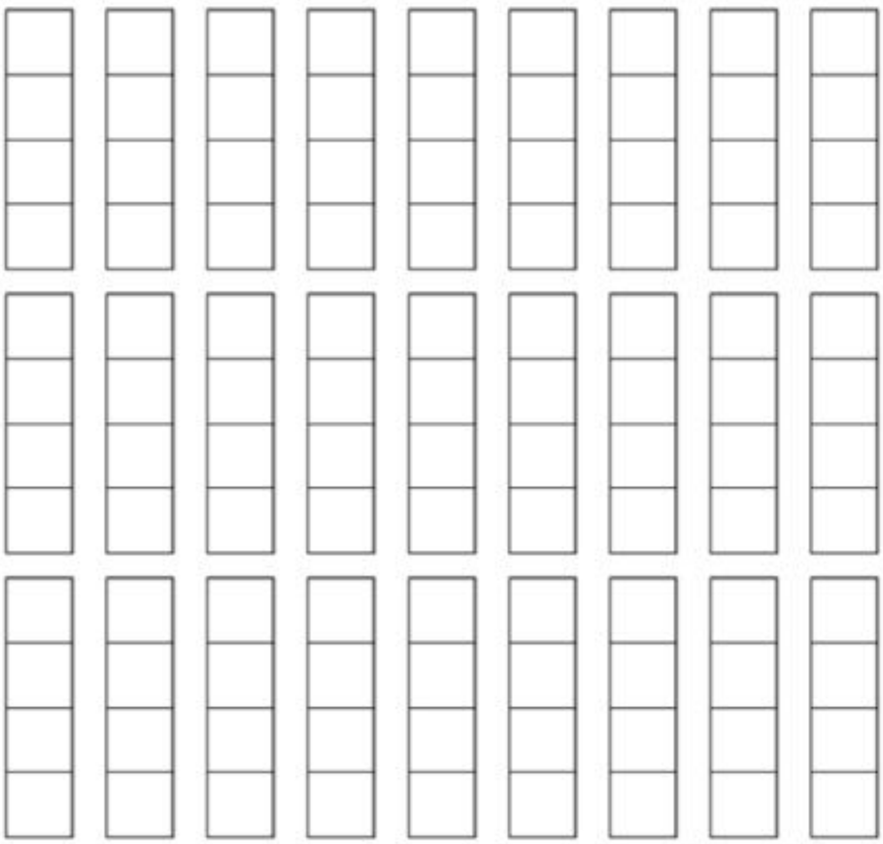
Allow some time for different learners to suggest ideas. Encourage any sort of rearranging which helps to make the pattern clear. If no-one suggests it, point out that we can group them in different ways - with red at the top, yellow at the top or blue at the top, etc. and that this helps us to see if there are any missing.

Then each pair can repeat the activity to convince themselves that they have found all possible towers.

Suggestions for Teaching UPPER PRIMARY

When the learners have done the activity above you might suggest that they work in groups of four to make as many four-block towers as possible, arranging them as they work so that they can be sure that they have found them all. You could give out strips of coloured squares if you are likely to run out of multilink. Cutting out and rearranging is a powerful way of working systematically to organise thinking.





Use this sheet to record all the different towers you can make with blocks of 4 different colours.

Suggestions for Teaching SECONDARY

You might ask the following question:
"Suppose the class lines up in a single line one behind the other outside the door before coming into the classroom. How many different arrangements are there for the order that the learners stand in line and enter the classroom?"

Start by the learners discussing this problem in pairs and trying to work out how to solve it, then have a class discussion. **Starting with simple cases is one of the best problem solving techniques.** If none of the class suggest doing this you can ask how many arrangements of the line there would be for 2 learners and then 3 learners and then longer lines of 4 learners and 5 learners and so on. This is of course exactly the same problem as the towers activity and you could give out multilink cubes or squares of coloured paper to help learners to work out the simple cases.

Key questions

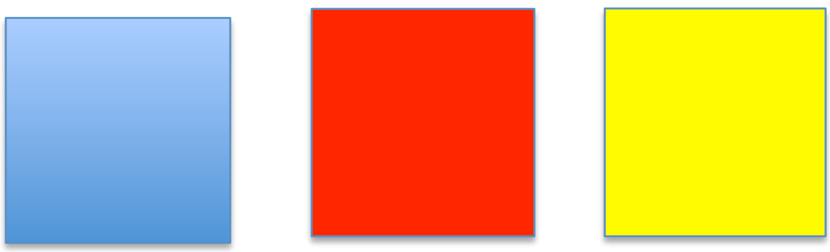
Have we got them all? How can we be sure?
How could we rearrange them to help?

Possible extension

Working with four blocks is an extension to the 3-block problem, but as an added extension you could again ask if the learners are sure that they haven't missed any out, and how they know. Look for different ways of arranging and listen for the explanations that go with them. Older or very able learners should work towards understanding the general number pattern and finding the formula for factorial n.

Possible support

Learners who struggle could be encouraged to cut out the three squares below and rearrange them in different orders.



Look for other similar problems, such as dressing in blue, red or yellow t-shirts on Monday, Tuesday and Wednesday, which help to reinforce the importance of working systematically.