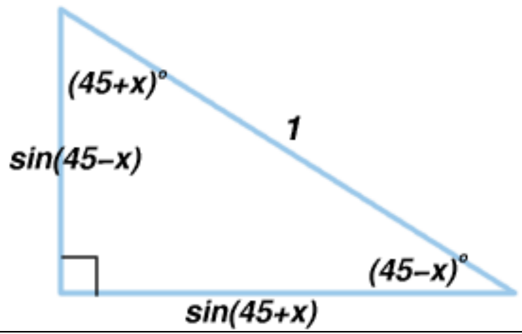


DEGREE CEREMONY

Draw a triangle with angles *x*, (45 + *x*) and (45 – *x)* degrees.

What does Pythagoras Theorem tell you about these angles?

Use this information to find this sum of squares of sines:

$\sin^2 1^\circ + \sin^2 2^\circ + \dots + \sin^2 359^\circ + \sin^2 360^\circ$

Many thanks to NRICH for this problem.

SOLUTION

From the definition of sine we know that, in a right angled triangle with hypotenuse 1 unit, the lengths of the sides are given by the sines of the opposite angles as shown in the diagram.

From Pythagoras Theorem we get $\sin^2(45 - x) + \sin^2(45 + x) = 1$ for all values of x from 1 to 44 so adding pairs of values that add up to 1 we get:

$\sin^2 1^\circ + \sin^2 89^\circ = 1$
$\sin^2 2^\circ + \sin^2 88^\circ = 1$
…
$\sin^2 43^\circ + \sin^2 47^\circ = 1$
$\sin^2 44^\circ + \sin^2 46^\circ = 1$

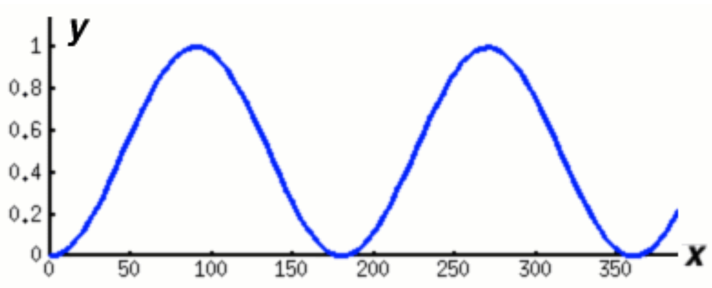
Using these identities, not forgetting $\sin^2 45^\circ$, we get:

$\sin^2 1^\circ + \sin^2 2^\circ + \dots + \sin^2 44^\circ + \sin^2 45^\circ + \sin^2 46^\circ + \dots + \sin^2 88^\circ + \sin^2 89^\circ$

$= (\sin^2 1^\circ + \sin^2 89^\circ) + \dots + (\sin^2 44^\circ + \sin^2 46^\circ) + \sin^2 45^\circ$

$= 44 + (1/\sqrt{2})^2 = 44\frac{1}{2}$

From the symmetry of the sine graph we have sin(90+x) = sin(90-x) and sin (180 + x) = - sin x.



As we see from the graph of $y = \sin^2 x$

$\sin^2 1 = \sin^2 179^\circ = \sin^2 181^\circ = \sin^2 359^\circ$
$\sin^2 2 = \sin^2 178^\circ = \sin^2 182^\circ = \sin^2 358^\circ$
$\sin^2 3 = \sin^2 177^\circ = \sin^2 183^\circ = \sin^2 357^\circ$
…
$\sin^2 89 = \sin^2 91^\circ = \sin^2 269^\circ = \sin^2 271^\circ$

So the sum of the squares of sines from 1° to 360°, not forgetting 90°, 180° and 270° is given by:

$$\sum_{\theta=1}^{\theta=360} \sin^2\theta = 4\left(\sum_{\theta=1}^{\theta=89} \sin^2\theta\right) + \sin^2 90 + \sin^2 180 + \sin^2 270$$
$$= 4(44.5) + 1 + 0 + 1$$
$$= 180$$

NOTES FOR TEACHERS

Why do this activity?

This activity gives learners practice in working with the sine function and using its properties. It is a non standard type question and it makes a change from using the periodicity and symmetries of the sine function in solving trig equations and it gives perhaps a surprising result and, many would say, a pretty one.

Intended learning outcomes

Development of familiarity with the graph of the sine function, its periodicity and the reduction formulas.

Possible approach

You might start by asking the learners to sketch the graph of $y = \sin^2 x$ from 0 to 360 and then having a class discussion about it. This should lead naturally to the learners noticing that they have to sum the series from 0 to 89 and then multiply by 4. It takes a bit more thought to realise that the value of $\sin^2 90$ has to be considered separately but at this stage in the lesson you might ask the learners to use what they have been discussing to find the sums of the squares of the sines from 1 to 360.

Learners will gain very little if the teacher shows them how to find the solution. It is important that learners have time to play with the ideas for themselves and they need the opportunity to think for themselves. You might want to ask the learners to work individually for a time. Then, to help the learners who struggle and to check answers it usually helps for learners to work in pairs after a while and to discuss their answers and reasons with a partner.

Some learners may come up with the answer 4x44 = 176 and others may jump to the conclusion that the answer is 180 without spotting that $\sin^2 45$, $\sin^2 135$, $\sin^2 225$ and $\sin^2 315$ have to be considered separately. You can ask pair of learners to explain their solution and try to check that everyone has understood.

Then summarize what they should know about the sine function.

Key questions

What does the graph of y=sinx look like? What about its period?
What do you notice about the graph of $y = \sin^2 x$? What about its period?
What does Pythagoras Theorem tell you about the angles (45+x) and (45-x)? Which values can x take?
Have you included all the whole number values of x from 1 to 360?
Have any values of x been left out?

Possible extension

Learners might work out the similar identity for $\cos^2 x$