

AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES SCHOOLS ENRICHMENT CENTRE (AIMSSEC)

AIMING HIGH



HELP

What does Pythagoras Theorem tell you about:

 $sin^{2}1^{\circ} + sin^{2} 89^{\circ}$ and $sin^{2}44^{\circ} + sin^{2} 46^{\circ}$ and other similar sums of sin² for angles between 1° and 89°. Now use this information to find part of the sum of squares required and use your knowledge of the values of the sine function for other angles to get the answer.

NEXT

Look for a similar identity for $\cos^2 x$

NOTES FOR TEACHERS

SOLUTION

From the definition of sine we know that, in a right angled triangle with hypotenuse 1 unit, the lengths of the sides are given by the sines of the opposite angles as shown in the diagram.

From Pythagoras Theorem we get $\sin^2(45 - x) + \sin^2(45 + x) = 1$ for all values of x from 1 to 44 so adding pairs of values that add up to 1 we get:

 $sin^{2}1^{\circ} + sin^{2} 89^{\circ} = 1$ $sin^{2}2^{\circ} + sin^{2} 88^{\circ} = 1$... $sin^{2}43^{\circ} + sin^{2} 47^{\circ} = 1$ $sin^{2}44^{\circ} + sin^{2} 46^{\circ} = 1$

Using these identities, not forgetting $\sin^2 45^\circ$, we get:

 $\sin^2 1^\circ + \sin^2 2^\circ + \dots + \sin^2 44^\circ + \sin^2 45^\circ + \sin^2 46^\circ + \dots + \sin^2 88^\circ + \sin^2 89^\circ$

 $= (\sin^2 1^\circ + \sin^2 89^\circ) + \dots + (\sin^2 44^\circ + \sin^2 46^\circ) + \sin^2 45^\circ$

$$= 44 + (1/\sqrt{2})^2 = 44\frac{1}{2}$$

From the symmetry of the sine graph we have sin(90+x) = sin(90-x) and sin(180 + x) = -sin x.



As we see from the graph of $y = \sin^2 x$ $\sin^2 1 = \sin^2 179^\circ = \sin^2 181^\circ = \sin^2 359^\circ$ $\sin^2 2 = \sin^2 178^\circ = \sin^2 182^\circ = \sin^2 358^\circ$ $\sin^2 3 = \sin^2 177^\circ = \sin^2 183^\circ = \sin^2 357^\circ$

$$\sin^2 89 = \sin^2 91^\circ = \sin^2 269^\circ = \sin^2 271^\circ$$

So the sum of the squares of sines from 1° to 360°, not forgetting 90°, 180° and 270° is given by:

$$\sum_{\theta=1}^{\theta=360} \sin^2 \theta = 4(\sum_{\theta=1}^{\theta=89} \sin^2 \theta) + \sin^2 90 + \sin^2 180 + \sin^2 270$$

= 4(44.5) + 1 + 0 + 1
= 180

Diagnostic Assessment This should take about 5-10 minutes.

- 1. Write the question on the board, say to the class:
- "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
- 2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
- 3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
- 4. Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers. It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.
- 5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.



Why do this activity?

This activity gives learners practice in working with the sine function and using its properties. It is a non standard type question and it makes a change from using the periodicity and symmetries of the sine function in solving trig equations. Also it gives perhaps a surprising result and, many would say, a pretty one.

Learning objectives

In doing this activity students will have an opportunity to develop familiarity with the graph of the sine function, its periodicity and the reduction formulas.

Generic competences

In doing this activity students will have an opportunity to:

- think mathematically, reason logically and give explanations and proofs;
- think flexibly, be creative and innovative and apply knowledge and skills.

Suggestions for teaching

Learners will gain very little if the teacher shows them how to find the solution. It is important that learners have time to play with the ideas for themselves and they need the opportunity to think for themselves. You might want to ask the learners to work individually for a time. Then, to help the learners who struggle and to check answers it usually helps for learners to work in pairs after a while and to discuss their answers and reasons with a partner.

You might start the lesson by asking the learners to sketch the graph of $y = \sin^2 x$ from 0 to 360 and then having a class discussion about it. This should lead naturally to the learners noticing that they have to sum the series from 0 to 89 and then multiply by 4. It takes a bit more thought to realise that the value of $\sin^2 90$ has to be considered separately but at this stage in the lesson you might ask the learners to use what they have been discussing to find the sums of the squares of the sines from 1 to 360.

Some learners may come up with the answer 4x44 = 176 and others may jump to the conclusion that the answer is 180 without spotting that $\sin^2 45$, $\sin^2 135$, $\sin^2 225$ and $\sin^2 315$ have to be considered separately. You can ask pair of learners to explain their solution and try to check that everyone has understood.

Then summarize what they should know about the sine function.

Key questions

- What does the graph of y=sinx look like? What about its period?
- What do you notice about the graph of $y = \sin^2 x$? What about its period?
- What does Pythagoras Theorem tell you about the angles (45+*x*) and (45-*x*)? Which values can *x* take?
- Have you included all the whole number values of x from 1 to 360?
- Have any values of *x* been left out?

Follow up

The Sine Problem <u>https://nrich.maths.org/436</u> Tangled Trig Graph <u>https://aiminghigh.aimssec.ac.za/years-10-12-tangled-trig-graphs/</u>

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. New material will be added for Secondary 6. The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is beyond the school curriculum for Grade 12 SA. For resources for teaching A level mathematics see https://nrich.maths.org/12339

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	Lower Primary	Upper Primary	Lower Secondary	Upper Secondary
	or Foundation Phase			
	Age 5 to 9	Age 9 to 11	Age 11 to 14	Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6