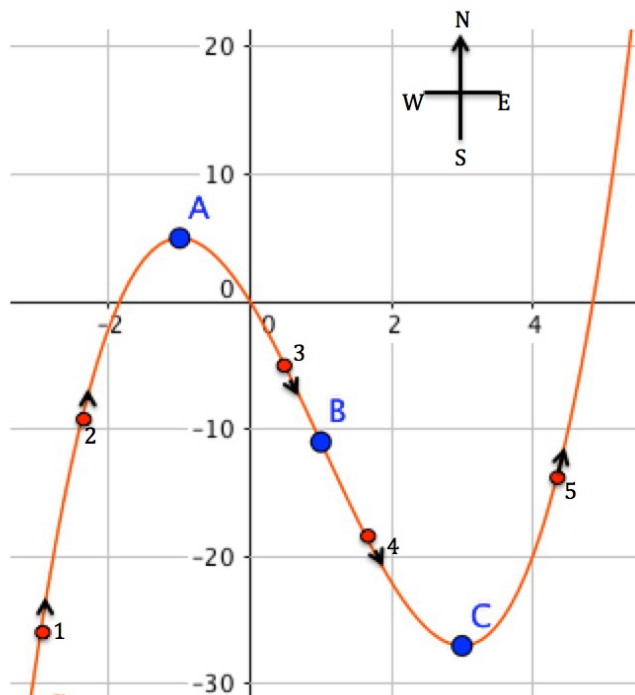


D'RIVING



Imagine this graph is a road and you are driving from the bottom left hand (South West) corner to the top right hand (North East) corner. Describe how you would steer the car.

When would you be turning the steering wheel to the right?

When would you be turning the steering wheel to the left?

When would you be driving straight ahead?

Now look at the graph again. How sharp were those bends? Thinking not of the graph, what was happening to the gradient when you were steering to the right?

What was happening to the gradient when you were steering to the left.

What was happening to the gradient when you were steering straight ahead?

What was the gradient when you were travelling due East?

This is the graph of $f(x) = x^3 - 3x^2 - 9x$.

Find the first derivative and the second derivative of this function and the coordinates of points A, B and C.

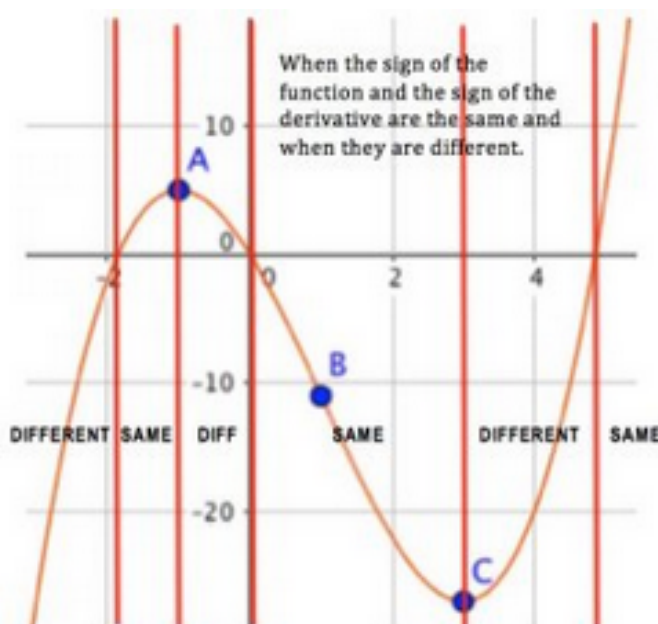
When do this cubic function and its first derivative have the same sign?

Sketch the graph of a cubic with only one turning point.

Sketch graphs of straight lines and quadratic functions.

Describe when these functions and their first derivatives have the same sign and when they have different signs.

	1	2	A	3	B	4	C	5
Gradient	+	+	0	-	-	-	0	+
First derivative	+	+	0	-	-	-	0	+
Gradient increasing I or decreasing D	D	D	D	D	*	I	I	I
Second derivative	-	-	-	-	0	+	+	+



HELP

Visualisation is the key to advanced mathematical thinking. It engages different parts of the brain in making connections. You might visualise your car actually drawing the curve. As you describe how you would steer the car along the path, you are describing how to draw the curve.

NEXT

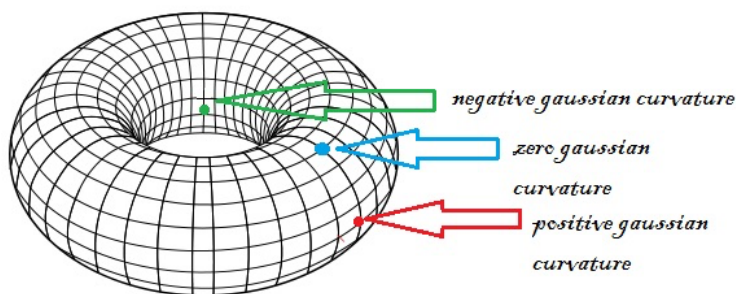
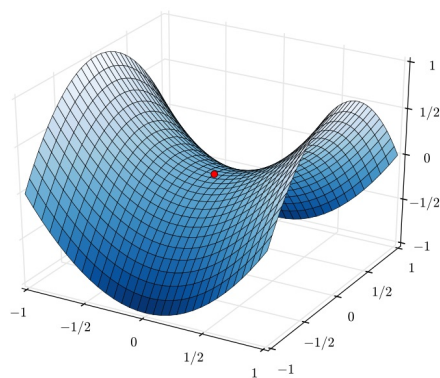
Describe how you would drive over this mountain pass in Lesotho.

See the blue image below for the mathematical view of a saddle point.



Try to create analogies for other concepts in calculus. This is very open ended but will really get you thinking about calculus as the mathematics of rates of change.

If you go on studying maths, and that means a lot more calculus, you'll learn about curvature of surfaces. A sphere has positive curvature. A flat surface has zero curvature. Some surfaces, like a saddle and a torus have both positive and negative curvature at different points of the surface.



Sign of the derivative of a function at each point

The derivative of the function is positive when travelling towards the north, negative when travelling towards the south.

Sign of the second derivative of a function at each point

If your steering wheel is turned clockwise from the neutral position then the second derivative is negative. If it is turned anticlockwise from the neutral position then the second derivative at that point is positive.

Sign of the third derivative of a function at each point

If the steering wheel is in the process of turning in the anti clockwise direction then the third derivative is positive. If the steering wheel is in the process of turning in the clockwise direction then the third derivative is negative.

Differentiability condition at each point

The function is differentiable at points on the road when it is possible to drive along smoothly without having to suddenly turn the steering wheel.

Points of inflection

Points of inflection occur at the points, and only the points, where the steering wheel passes through the neutral position.

Note on terminology

The '**neutral position**' is the position of the steering wheel in which the car travels forwards in a straight line. A **clockwise turn** from this position causes the car to turn right and an **anticlockwise turn** from this position causes the car to turn left.

Cards with statements to discuss in relation to the D'RIVING graph.

Do you agree or disagree with these statements?

Give reasons for your answers.

GUIDE FOR HOMELEARNING

SOLUTION

You turn the steering wheel to the **right** from the start all the way to B because the gradient is **decreasing**.

You turn the steering wheel to the **left** all the way from B to the end because the gradient is **increasing**.

You drive **straight ahead** at B where the gradient is stays the same for an instant.

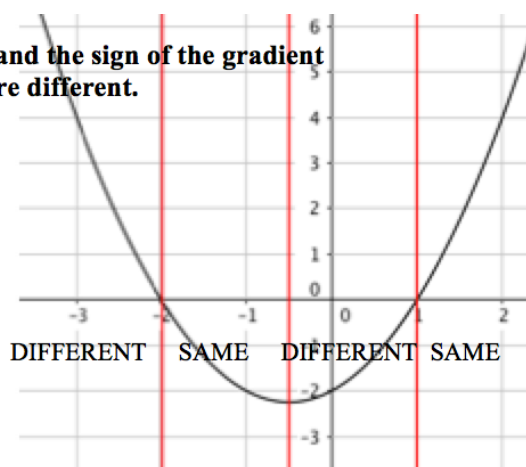
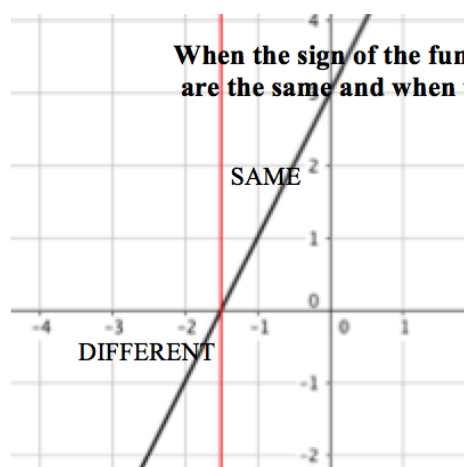
When travelling due East (in the positive x direction) the gradient is zero.

$$f(x) = x^3 - 3x^2 - 9x$$

The first derivative $f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$
so the coordinates of the turning points A and C where the first derivative is zero are A(-1, 5) and C(3, -27).

The second derivative $f''(x) = 6x - 6$
so the coordinates of B where the second derivative is zero are B(1, -11).

This cubic function and its first derivative have the same sign when the function is positive and the gradient is positive and also when the function is negative and the gradient is negative.



The vertical red lines separate the regions in which the sign of the function and the sign of the gradient are the same and when they are different.

Why do this activity?

This activity often proves incredibly useful to learners because having a sound geometrical visualisation for concepts in calculus is essential in any application. It also proves very useful in checking that calculations make sense and for uncovering misconceptions about calculus.

Learning objectives

In doing this activity students will have an opportunity to

- deepen understanding of functions, derivatives, turning points and gradients;
- explore the relationship of signs of functions and derivatives to gradients.

Generic competences

In doing this activity students will have an opportunity to:

- **think flexibly**, be creative and innovative and apply knowledge and skills;
- **visualize** and develop the skill of interpreting and creating visual images to represent concepts and situations.

Suggestions for Home Learning

You might like to use the cards on page 2 so that you have clear statements to work with.

This need not be a long activity and can be used at any point in the curriculum where the concepts in any of the 5 analogies on page 2 have been encountered. You can focus on a couple of the most relevant analogies if desired.

It might help to think about the ideas in a small groups and sketch 'road maps' on which to test your ideas.

Alternatively, you can sketch a curve with, say, 4 turning points and ask yourself about the motion of the imaginary steering wheel as you trace your finger along the curve. You could record the motion of the steering wheel, paying particular attention to the direction or speed of turn. You could then sketch a more 'demanding' road and repeat the exercise.

There are at least three levels of approach to this problem:

- 1) Once you are clear as to which analogies are largely reliable you can move on. The analogies can be referred to as a guide throughout subsequent study of calculus.
- 2) You can try to construct convincing justification that the analogies are sound, including some thought on when the analogies break down (i.e. what sorts of roads do the analogies work for, and what sorts of 'pathological' roads do the examples not work for?)
- 3) You might try to come up with some analogies of your own which others might test out. For example, other analogies for the sign of the gradient might involve mountains, valleys or hills.

Note that various misconceptions might be unearthed during this task, and many more advanced concepts in mathematics might be raised. Wikipedia is often a good place to find out more; it's not always 100% accurate but generally it's OK and at least some of the content is easy enough for you to understand.

Misconceptions or errors to look out for are:

1. The steeper the gradient the more the wheel needs to be turned
2. A function can be used to describe, say, a circle (No: A function is single valued)
3. A point of inflection must also be a stationary point (No: That is a stationary point of inflection)

Advanced concepts in mathematics which might be raised in some form are:

1. What is a function as opposed to a curve?
2. What is a continuous / differentiable function?
3. What is curvature?
4. Are there functions that are only twice differentiable?

Key questions

- Who can describe the motion of a wheel through a journey?
- Can you imagine driving along the road indicated on this map?
- For what sorts of crazy curves might these analogies not work?
- Can you give a clear justification for your results (using words or algebra)?
- What can we say about a car which is moving due north at some point?

Diagnostic Assessment

This should take about 5–10 minutes at the end of the session.

If you are working with a group, the leader can show this question and say:

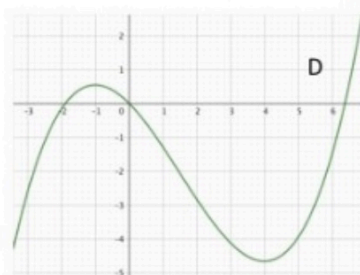
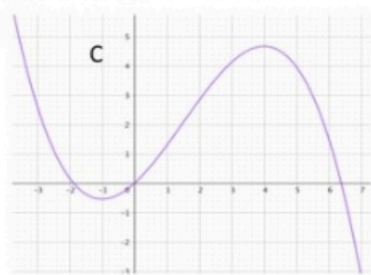
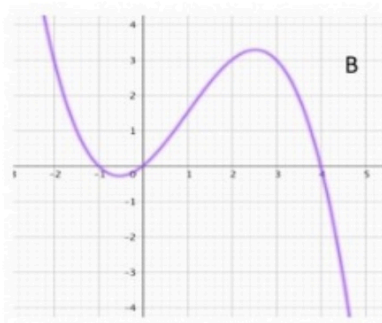
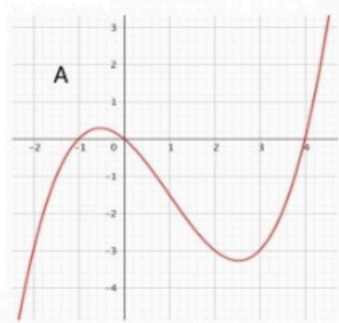
“Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D”.

Sketch the graph of a function f of which the derivative function f' satisfies the following conditions:

$$f'(x) = 0 \text{ for } x = -1 \text{ and for } x = 4,$$

$$f'(x) < 0 \text{ for } x < -1 \text{ and for } x > 4,$$

$$f'(x) > 0 \text{ for } -1 < x < 4.$$



The correct answer is C. Local maximum or minimum at $x = -1$ and $x = 4$
for $x < -1$ and $x > 4$ $f(x)$ is decreasing
and for $-1 < x < 4$ $f(x)$ is increasing

For answers A, B and D students have misread the graphs looking at intercepts with x-axis instead of turning points and the sign of the function instead of the sign of the derivative. <https://diagnosticquestions.com>

- 1). Notice how the students respond. Ask them to explain why they gave their answers and DO NOT say whether it is right or wrong but simply thank the student for giving the answer.
- 2). It is important for learners to explain the reason for their answer. Often by giving an explanation you come to understand the idea better yourself.
- 3). Try to listen carefully to these reasons and try to decide if your own answer was right or wrong.
- 4). Vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.

For younger learners

If you have a computer connected to the internet then download for free the brilliant easy-to-learn software MSW Logo <https://mswlogo.en.softonic.com/download>. Written about 50 years ago as an educational tool, nothing better has been created since. The principle of steering along different paths is what underlies Logo.

See First Forward Into Logo for a series that introduces Logo programming for beginners. The [12 challenges](#) are written for you to work through at your own pace. You don't need to know anything about coding (programming) at the start: you will learn enough at the first stage to write your own programs, and you may surprise yourself by how much you can achieve right away.

The idea of driving along curves works well for all ages and older students can help the younger children in their households. Small children, even pre-schoolers, can gain a lot from exploring different curves by imagining driving a car along the curves. Very young children can drive 'matchbox' type toy cars; draw a road system for them on some newspaper.

Primary age children can explore 2D shapes by imagining that they are driving a car along the edges of the shapes. If they are exploring polygons they can think about the angles they need to turn at the corners.

Secondary age learners can enjoy exploring the properties of curves drawn on coordinate axes.

Follow up

Derivative Matching <https://aiminghigh.aimssec.ac.za/year-12-derivative-matching/>