GP ALGEBRAICALLY

Multiply out these expressions:
What do you notice?
Does this pattern continue?
Try a few more steps.
If you think that the pattern continues, can you prove it?

$(1+r)(1-r)$,
$(1+r+r^2)(1-r)$,
$(1+r+r^2+r^3)(1-r)$,
$(1+r+r^2+r^3+r^4)(1-r)$,
$\cdots$

Why does this pattern show that, for $r \neq 1$, the sum of the powers of r from 0 to (n -1) is given by this formula

$$\sum_{i=0}^{n-1} r^i = 1 + r + r^2 + r^3 + \ldots + r^{n-1} = \frac{1-r^n}{1-r}?$$

For -1 < r < 1 what happens to r^n as n gets bigger? What can you say about $\lim_{n\to\infty} r^n$

What does this suggest to you about the infinite sum of the geometric series:
$\sum_{i=0}^{\infty 1} r^i = 1 + \mathrm{r} + r^2 + r^3 + \cdots$ for -1 < r < 1?

HELP

Writing out the multiplications in full will help you to see the pattern.

NEXT

GP Geometrically https://aiminghigh.aimssec.ac.za/grade-12-gp-geometrically/

NOTES FOR TEACHERS

SOLUTION

$(1+ r)(1 - r) = 1 - r^2$

$(1+ r + r^2)(1 - r) = 1 - r^3$

$(1+ r + r^2 + r^3)(1 - r) = 1 - r^4$

$(1+ r + r^2 + r^3 + r^4)(1 - r) = 1 - r^5$

Here the sum of the series of increasing powers of r up to r^{n-1} multiplied by $(1 - r)$ always gives $1 - r^n$.

This happens because, in working out the product, alternate terms cancel out, and we are left with two remaining terms.

This can be proved rigorously using mathematical induction (not on the school syllabus in some countries and not required in this question).

From the product $(1+ r + r^2 + r^3 + \ldots + r^{n-1})(1 - r) = 1 - r^n$ it follows, by dividing both sides by $(1 - r)$, that for $r \neq 1$, the sum to n terms of the geometric series is given by the formula:

$$\sum_{i=0}^{n-1} r^i = 1 + r + r^2 + r^3 + \ldots + r^{n-1} = \frac{1 - r^n}{1 - r}.$$

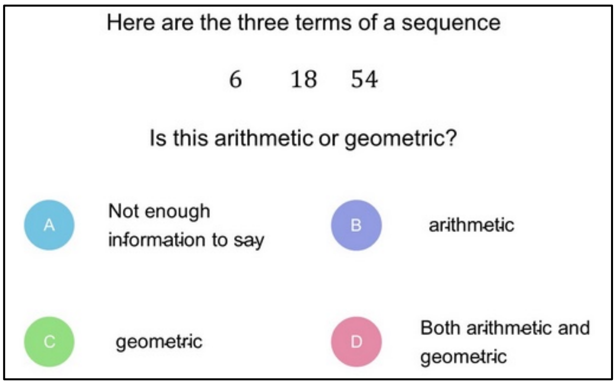
We next think about what happens if the series is infinite. Now $\lim_{n\to\infty} r^n = 0$ for $-1 < r < 1$, so from the above formula we get the sum of the infinite geometric series $\sum_{i=0}^{\infty 1} r^i = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$ for $-1 < r < 1$.

Why do this activity?

All students should approach geometric series in this way so that they can discover the formula for the sum of n terms for themselves. This is a very simple exercise in multiplying algebraic expressions to remove brackets, it is a pleasing result when you see it for the first time, and teachers should not rob their students of the joy of making their own discoveries. In the same way students can find out for themselves what happens to the formula in the extension to the infinite series. The skill on the part of the teacher is asking the right questions.

Diagnostic Assessment This should take about 5–10 minutes.
Write the question on the board, say to the class:
"Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".

1. Notice how the learners respond. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
2. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
3. It is important for learners to explain the reason for their answer to practise communication and thinking mathematically.



4. **Ask the class to vote again for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.**
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

The correct answer is C if this sequence has only 3 terms because it is then geometric with common ratio 3:
6 6×3 6×3^2
A. If these were the first three terms of a longer sequence then there is not enough information to say what series has these first three terms, for example the given terms could be terms of the sequence: $f(n) = 12n^2 - 24n + 18$. Just substitute
$n = 1, 2$ and 3 and you will get the values 6, 18 and 54.
B. The sequence is not arithmetic as the differences between the terms are not equal.
D. Constant sequences like c c c c ... are both arithmetic and geometric for all values of c (can you prove this?) and the sequences 0 0 0 0 ... is geometric for common ratios of any value.
https://diagnosticquestions.com

Intended learning outcomes

To know and understand the derivation of the formula for the sum of a finite geometric series and also for the sum of an infinite geometric series.

Generic competences

We need to prepare children for a job market where existing knowledge and skills have limited value unless they can be applied in novel ways to produce new knowledge that solves today's complex problems to improve the quality of life for all.
In doing this activity students will have an opportunity to:

- think mathematically and to reason logically;
- analyze, reason and record ideas effectively.

Suggestions for teaching

You may decide to concentrate for a while on the sum of a finite series. After following the teaching method recommended above **to enable your students to discover for themselves** the formula for the sum of a geometric series, you may get the learners to do some practice exercises. Before introducing the infinite geometric series you may decide to do further work on finite series, for example the generalisation to the formula for the series with a as the first term:

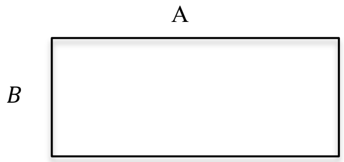
$$\sum_{i=0}^{n-1} ar^i = a + ar + ar^2 + ar^3 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

There are good reasons for considering only finite series. However, if the students can spot for themselves the pattern resulting from taking more and more terms of the series and multiplying by $(1 - r)$, then they will have a good understanding of the derivation of the formula. It will then be a natural further generalisation to consider the infinite series and it makes it an even bigger and more exciting discovery for the students.

Key questions

- Can you see a pattern? Can you describe it?
- What do you notice?
- Does this pattern continue?
- Can you prove it continues in the same way?
- Why do we only get two terms when we are multiplying a lot of terms in that bracket by $(1 - r)$?
- What happens when we divide by $(1 - r)$? Why do we have to exclude $r = 1$?
- Why are we saying this series has n terms when the highest power is $(n - 1)$?
- What happens to of r^n as n gets bigger and bigger if r is between -1 and +1?
- Why are we only talking about values of r between -1 and +1 when we think about the limit of r^n as n tends to infinity?

Follow up



Why are the series called *geometric* and *arithmetic*?
The name geometric series arises from the geometric problem of finding the dimensions of a square having the same area as a rectangle.

The square with edges of length $\sqrt{(AB)}$ has the same area as the rectangle with edges A and B.

The terms A, $\sqrt{(AB)}$ and B are in a **geometric sequence** with ***common ratio*** $\sqrt{\frac{B}{A}}$

because $A \times \sqrt{\frac{B}{A}} = \sqrt{AB}$ and

$$\sqrt{AB} \times \sqrt{\frac{B}{A}} = B.$$

We call $\sqrt{(AB)}$ the ***geometric mean*** of A and B.

Each term in a ***geometric sequence*** is the geometric mean of the term before and the term after it.

The terms A, $½(A + B)$ and B are in an **arithmetic sequence with *common difference***

$½(B - A)$

because $½(A + B) - A = B - ½(A + B) = ½(B - A).$

We call $½(A + B)$ the ***arithmetic mean*** of A and B.

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Also see: GP Geometrically https://aiminghigh.aimssec.ac.za/gp-geometrically/

Go to the **AIMSSEC AIMING HIGH** website for lesson ideas, solutions and curriculum links: http://aiminghigh.aimssec.ac.za
Subscribe to the **MATHS TOYS YouTube Channel** https://www.youtube.com/c/MathsToys/videos
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Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and school years up to Secondary 5 in East Africa.
New material will be added for Secondary 6.
For resources for teaching A level mathematics (Years 12 and 13) see https://nrich.maths.org/12339
Mathematics taught in Year 13 (UK) & Secondary 6 (East Africa) is beyond the SA CAPS curriculum for Grade 12

	Lower Primary Approx. Age 5 to 8	Upper Primary Age 8 to 11	Lower Secondary Age 11 to 15	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13