AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE
TEACHER NETWORK

THE GREEDY ALGORITHM

Ancient Egyptians used only unit fractions, that is fractions with 1 as the numerator.
They wrote all fractions as the sum of unit fractions with different denominators.
We are going to explore how we can write the fractions we use in the form of Egyptian fractions using a method called the Greedy Algorithm invented by Fibonacci.

For example $\frac{2}{5}=\frac{1}{4}+\frac{1}{10}+\frac{1}{20}$

Can you write $\frac{5}{9}$ as the sum of unit fractions?

That is not very difficult but what did the Egyptians do with fractions that we write with large numerators such as $\frac{61}{66}$?

Using the Greedy Algorithm, so called because at each step you use the largest possible unit fraction that is smaller than the one you are working with, we see that the largest possible fraction smaller than $\frac{61}{66}$ is $\frac{1}{2}$ so the first step is $\frac{61}{66}-\frac{1}{2}=\frac{28}{66}=\frac{14}{33}$.

Then we subtract $\frac{1}{3}$ and get $\frac{14}{33}-\frac{1}{3}=\frac{3}{33}=\frac{1}{11}$ so $\frac{61}{66}=\frac{1}{2}+\frac{1}{3}+\frac{1}{11}$.

Now if you try this method on $\frac{7}{15}$ you should get $\frac{7}{15}=\frac{1}{3}+\frac{1}{8}+\frac{1}{120}$. Try it.

Next write $\frac{61}{84}$ as the sum of unit fractions.

Now choose some fractions of your own and see if you can make them into sums of unit fractions.
Does the greedy algorithm always work?
Can all fractions be expressed as a sum of different unit fractions by applying the Greedy Algorithm?
Can you explain why?

SOLUTION

$\frac{7}{15}-\frac{1}{3}=\frac{2}{15}$ and $\frac{2}{15}-\frac{1}{8}=\frac{16-15}{120}=\frac{1}{120}$ so, combining these results: $\frac{7}{15}=\frac{1}{3}+\frac{1}{8}+\frac{1}{120}$.

$\frac{61}{84}-\frac{1}{2}=\frac{19}{84}$ and $\frac{19}{84}-\frac{1}{5}=\frac{95-84}{420}=\frac{11}{420}$ and $\frac{11}{420}-\frac{1}{40}=\frac{22-21}{840}=\frac{1}{840}$ so, combining these results $\frac{61}{84}=\frac{1}{2}+\frac{1}{5}+\frac{1}{40}+\frac{1}{840}$.

Each step reduces the numerator of the remaining fraction. But we must prove that this will always happen.

Starting with x/y, we can find the two unit fractions 1/a and 1/(a−1) so that x/y lies between them:
$1/a \le x/y < 1/(a-1)$ so 1/a is the largest unit fraction that we can subtract from x/y.

Subtracting 1/a from x/y leaves us with (ax−y)/ay.

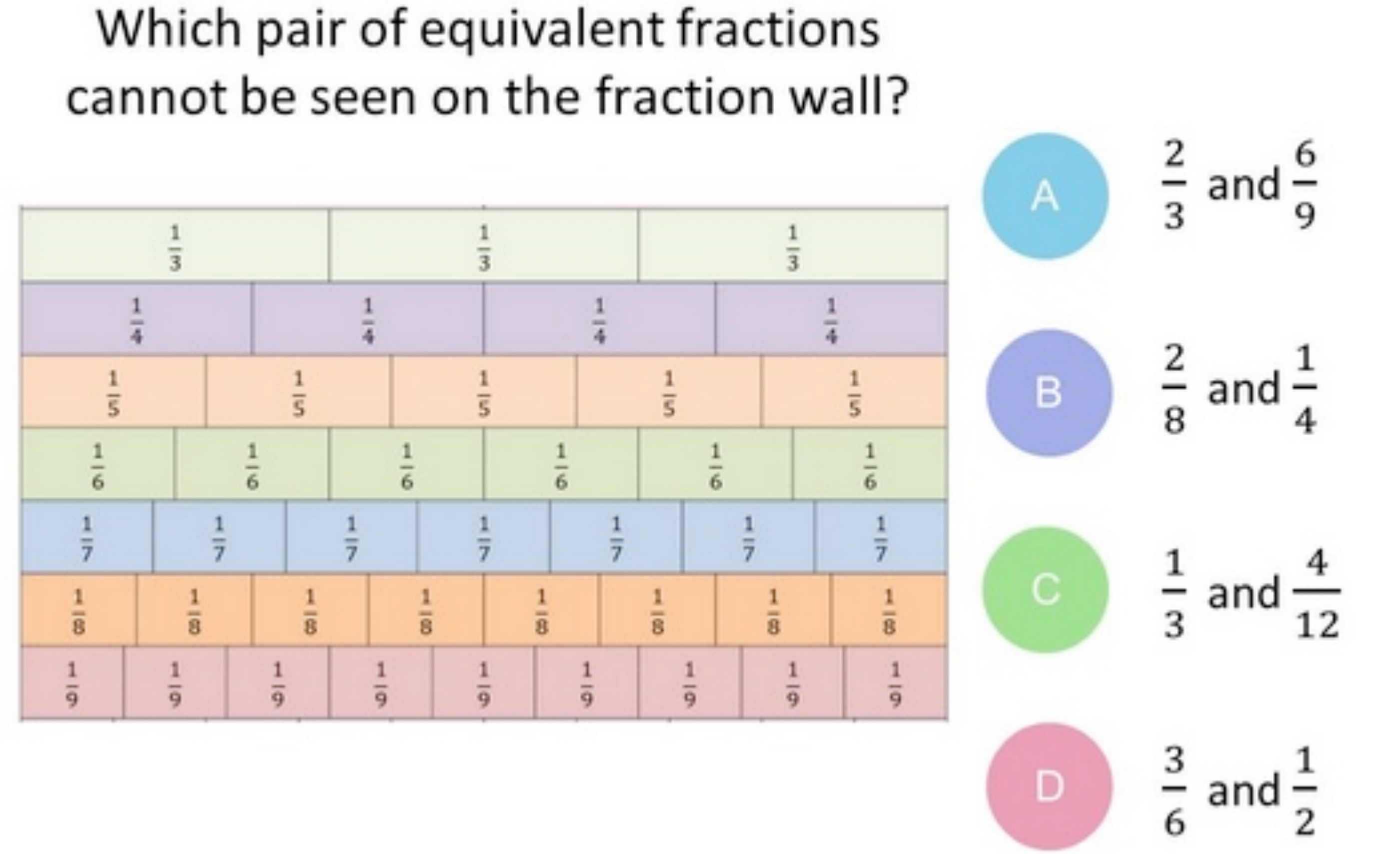
We know that $x/y < 1/(a-1)$ so it follows that $x(a-1) < y$
and so $ax-x < y$
therefore $ax-y < x$.
Since $ax-y < x$ it follows that (ax−y)/ay will have a smaller numerator than x/y
so when we subtract the largest possible unit fraction from x/y we will be left with a fraction with a smaller numerator.

As we repeat the process the numerator will get smaller and smaller until it eventually reaches 1 (when we will be left with a unit fraction).

Notes for teachers

Diagnostic Assessment This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.



C. is the correct answer.

Common Misconceptions

A or B. If this was not just a guess then the learner does not know how to use a fraction wall and will be helped by some work with this one.

D. Learners may pick D because this fraction wall does not show ½s but 3/6 can be matched to 2/4 or 4/8.

White Rose Maths Diagnostic Questions https://diagnosticquestions.com

Why do this activity?

This activity offers learners an opportunity to engage with some mathematical ideas in depth and not just with the rather mechanical process of adding and subtracting fractions. They will get practice in adding and subtracting fractions in an interesting context.

This activity requires learners to compare fractions and may deepen their understanding of their relative sizes.

The algebraic proof that the Greedy Algorithm always works will be too difficult for all but older and very talented learners but some of them will realise that the numerator decreases step by step so it must eventually get to zero. This in itself is a stage of mathematical reasoning that it is helpful to encourage.

Suggestions for Teaching

This activity follows on from the activity https://aiminghigh.aimssec.ac.za/grades-7-to-10-egyptian-fractions/

Ask learners to work in pairs, to decide on a fraction, and then to try to write it as an Egyptian fraction, that is as a sum of fractions that have 1 as the numerator and different denominators. Then ask some learners to write their examples on the board for other learners to check.

Ask learners who have not succeeded to write their examples on the board so that the whole class can try them. The class should share strategies. Do they always get the result quickly? Do different strategies give the same results?

If you start with 7/9, for example, and apply different strategies, you may end up with different answers, for example:
7/9 = 1/2 + 1/6 + 1/9 or
7/9 = 1/3 + 1/6 + 1/9 +1/15 +1/18 + 1/30 + 1/90
....
Introduce the Greedy Algorithm and ask if anyone has been using a strategy like it.

Ask learners to choose fractions of their own and to use the Greedy Algorithm.

Does it always work? Can they explain why it works?

Key Questions

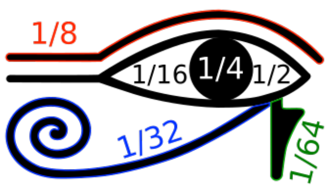
Is your fraction larger or smaller than 1/2? How do you know?
Is your fraction larger or smaller than 1/3? How do you know?
Is your fraction larger or smaller than 1/4? How do you know?
....

Possible extension

Learners can set challenges for each other.

They could try to prove why Fibonacci's Greedy Algorithm always terminates (the numerators get smaller and smaller and so they must get down to one).

Does the Greedy Algorithm always result in the expression with the smallest number of terms?
Can anyone find a counter example?



The fractions representing 1/2, 1/4, 1/8, 1/16, 1/32 and 1/64 were represented by parts of the Eye of Horus.

Ask learners to choose some fractions and to see how close they can get using only these five 'Eye of Horus' unit fractions.

Possible support

Learners could first do the activity: https://aiminghigh.aimssec.ac.za/grades-7-to-10-egyptian-fractions/

Learners who find difficulty in comparing the size of fractions might need to first do some work on equivalent fractions. It is important for them to understand that **increasing the denominator makes the fraction smaller.**