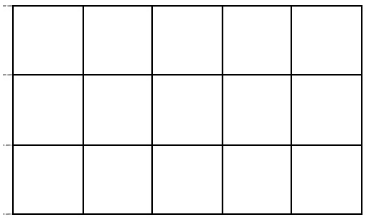
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

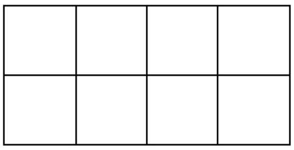
LATTICE POINTS

Lattice points are the intersections of the grid lines.

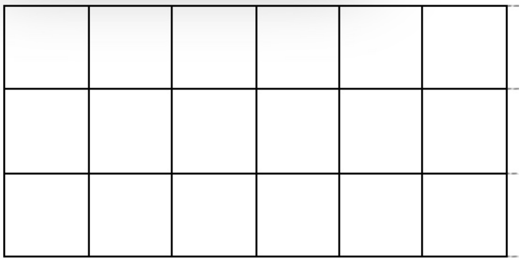
5 by 3



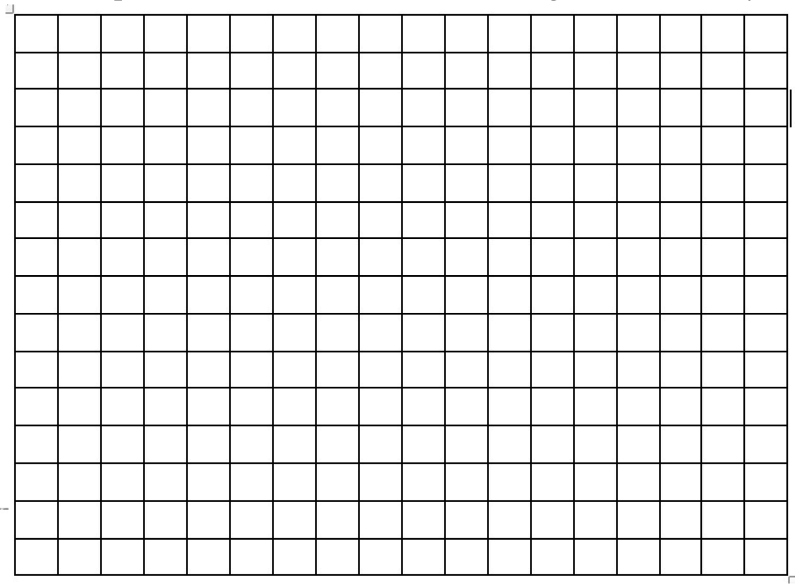
4 by 2



6 by 3



18 by15



FOR YOUNGER LEARNERS - Draw in diagonal lines corner to corner on these 4 grids. How many lattice points do your lines go through?

Draw another grid. Draw in the diagonal.
How many lattice points do your lines go through?

Draw more grids. Before you draw the diagonals, can you predict how many lattice points they will go through? How do you know?

FOR OLDER LEARNERS - A point whose x- and y-coordinates are both whole numbers is called a lattice point.

How many lattice points are there in the first quadrant (where both x and y coordinates are positive) that lie on the line $3x + 4y = 59$?

Find these points by different methods. How many methods can you find?

HELP

Carefully plot the graph of $3x + 4y = 59$.

One possible method is to find and join the intercepts with the x-axis and y-axis.

Notice that, when $x = 0$, $y = 59/4 = 14\frac{3}{4}$ and when $y = 0$, $x = 19\frac{2}{3}$.

These are not lattice points but they can be plotted and joined to give the straight line graph.

Then the lattice points can be read from the graph and checked numerically.

When you have found the lattice points by this method try to find an alternative method.

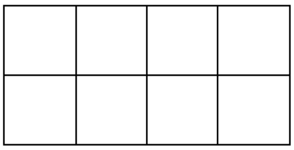
NEXT

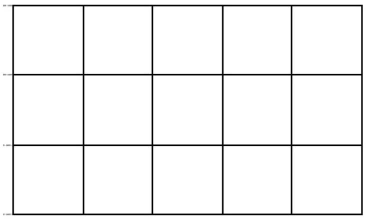
Find at least two methods (there are 4 different methods).

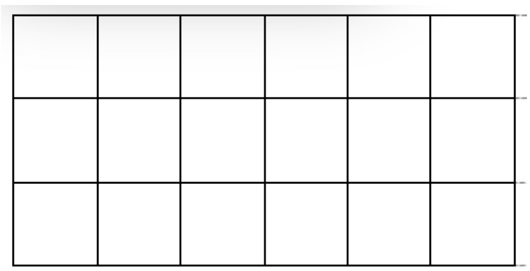
Say which you think is the best method and why.

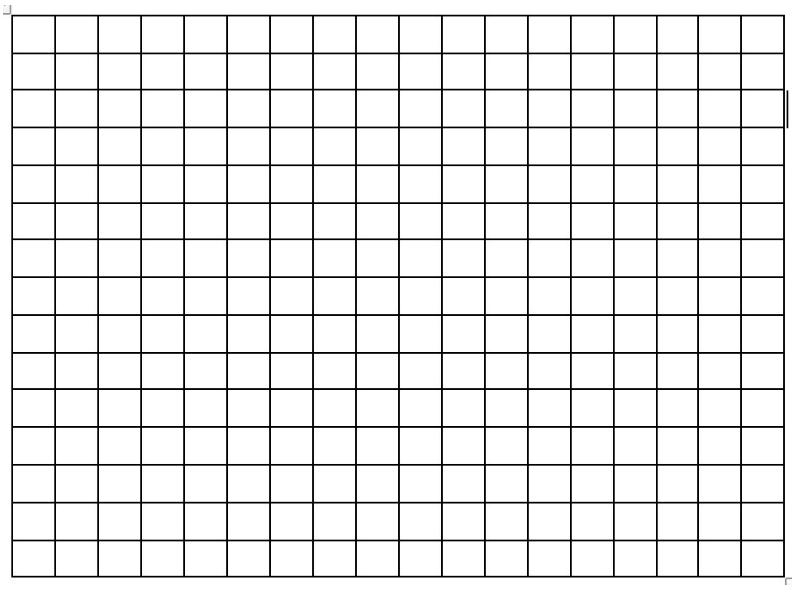
Use that method to find the lattice points in the first quadrant for 2x + 5y = 52?

Write a list of instructions for another learner to follow to use that same method for **other** examples.









HOME LEARNING GUIDE

SOLUTION

4 by 2 grid: The line goes through 1 interior lattice point.
The steps between lattice points are: 1 up and 2 across.

5 by 3 grid: The line goes through no interior lattice points as 5 and 3 have no common factor.

6 by 3 grid: The line goes through 2 interior lattice points.
The steps between lattice points are: 1 up and 2 across.

18 by 15 grid: The line goes through 2 interior lattice points.
The steps between lattice points are: 5 up and 6 across.
If this line is extended then the number patterns for the steps up and across are:
multiples of 5 up 5, 10, 15, 20, 25, 30,…
multiples of 6 across 6, 12, 18, 24, 30,…

METHOD 1 Using a table of values

Points in the first quadrant have x, y>0, so we could begin by substituting values into the equation of the line to check for lattice points on the line.

x	$3x$	$4y$	y
1	3	56	14
2	6	53	13.25
3			
4			
5			
…			

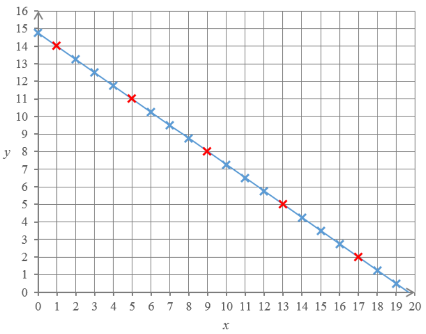
Using the table below, we can try positive whole numbers for x, and then find the value of $3x$. If x=1, then $3x$=3, so $4y$ must be 56 since 3+56=59, so y=56÷4=14. So the point (1,14) lies on the line and is a lattice point. This is shown on the first line of the table.

If x=2, then $3x$=6, so $4y$ must be 53 since 6+53=59, so y=53÷4=13.25. So the point (2,13.25) lies on the line, but it is not a lattice point.

Filling in the whole table will give all of the possible lattice points. Remember that y needs to be positive, so $3x$ must not be greater than 59, so x must be less than 20 (since 3×20=60>59). So the table only needs to go up to x=19.

x	$3x$	$4y$	y
1	3	56	14
2	6	53	13.25
3	9	50	12.5
4	12	47	11.75
5	15	44	11
6	18	41	10.25
7	21	38	9.5
8	24	35	8.75
9	27	32	8
10	30	29	7.25
11	33	26	6.5
12	36	23	5.75
13	39	20	5
14	42	17	4.25
15	45	14	3.5
16	48	11	2.75
17	51	8	2
18	54	5	1.25
19	57	2	0.5

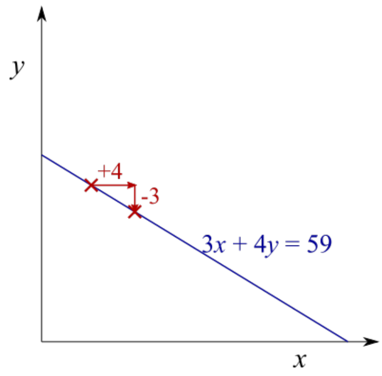
Counting the points in the table with whole number y coordinates, there are 5 lattice points. Plotting these points gives the graph below (with lattice points marked in red).



METHOD 2 Plotting the graph

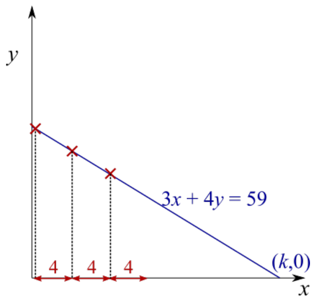
When x = 0, y = 59/4 = 14¾ and when y = 0, x = 19⅔.
These are not lattice points but they can be plotted and joined to give the straight line graph. Then the lattice points can be read from the graph and checked numerically.

METHOD 3 Using the gradient triangles
This is an improvement on METHOD 2
The gradient of the line is −¾. You can work this out by considering changes in x and y, or by rearranging $3x+4y=59$ into $y = -\frac{3}{4}x + 59/4$. This means that every time x increases by 4, y decreases by 3. This is shown on the diagram.



If you start at a lattice point, then there will be another lattice point 4 to the right and 3 down, and another 8 to the right and 6 down, and so on and also another lattice point 4 to the left and 3 up, and so on. So the lattice points, if there are any, will be evenly spaced along the x-axis at intervals of 4, as shown below – where the red crosses mark lattice points. That means that, if we can find a lattice point on the line between $x=0$ and $x=4$, then we can quickly find all the lattice points by adding 4 to the x coordinate until we get to k (where k is the x-intercept, as shown above).

However, if we can't find a lattice point between $x=0$ and $x=4$, then there won't be any lattice points at all, because lattice points should be at intervals of 4, so there can't be an interval of 4 that doesn't contain one.



If $x=1$, then $3x=3$, so $4y$ must be 56 because 3+56=59, so $y = 56÷4 = 14$. So the point (1,14) lies on the line and is a lattice point. Then there are also lattice points when x=5, 9, 13…. To work out how many there are in the first quadrant, we need to find the value of k, because if $x > k$ then the lattice point will not be in the first quadrant.

When $x=k$, $y=0$, and so $3×k + 4×0 = 59$, so $k=59/3=19⅔$. So the lattice points have x-coordinates 1, 5, 9, 13 and 17 - which means there are 5 lattice points.

METHOD 4 Using factorisation and properties of numbers
Suppose (a, b) is lattice point on the line $3x + 4y = 5\ 9$ in the first quadrant. Then a and b are positive whole numbers with $3a + 4b = 59$.
$3a + 4b$ can be written as $3(a+b) + b$. This is useful because, starting from $3(a+b) + b=59$, $3(a+b)=59-b$, so $59-b$ must be a multiple of 3.
Also, since a is positive, $3a$ is also positive, and so $4b<59$, so $b<59÷4$, so $b≤14$, since b is a whole number. That means that $59-b \geq 59-14$, so $59 - b \geq 45$.
So $59-b$ is a multiple of 3 that is between 45 and 59 so $59-b$ could be 45, 48, 51, 54 or 57. From each of those 5 options, we can find b, and then $a+b$, and then a. So there are 5 lattice points that satisfy the equation – so there are 5 lattice points on the line.

Why do this activity?

The activity uses the '**distances across and up'** features of the lines **that underpin the concepts and formulas of gradient and the distance between two points.** It is easy for learners to understand what they are trying to find in this activity – how many lattice points?

Learners of different ages can use grids of different sizes. They can use trial and improvement, or any one of several different methods to find the lattice points. Younger learners can just draw lines, count points and look for patterns of multiples and common multiples.

For older learners, the activity gives practice in applying what they know about equations of lines and gradients.

Intended learning outcomes

In doing this activity students will have an opportunity to:

- develop a deep understanding of the concept of gradient in mathematics;
- develop mathematical thinking and problem solving skills;
- gain a deeper understanding of linear functions and equations of straight lines.

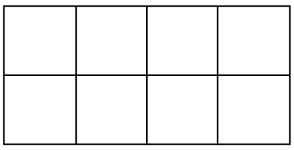
Generic competences

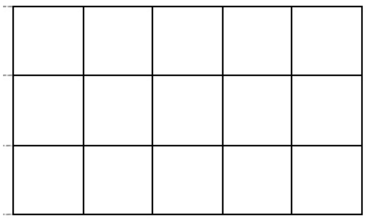
In doing this activity students will have an opportunity to:

- **think mathematically**, reason logically, interpret and **solve problems,** and give explanations and proofs;
- **think flexibly**, be creative and innovative - to apply knowledge and skills;
- **visualize** and develop the skill of interpreting and creating visual images to represent concepts.

Ideas for Home Learning

Young children





Count the squares.

How many across?

How many up?

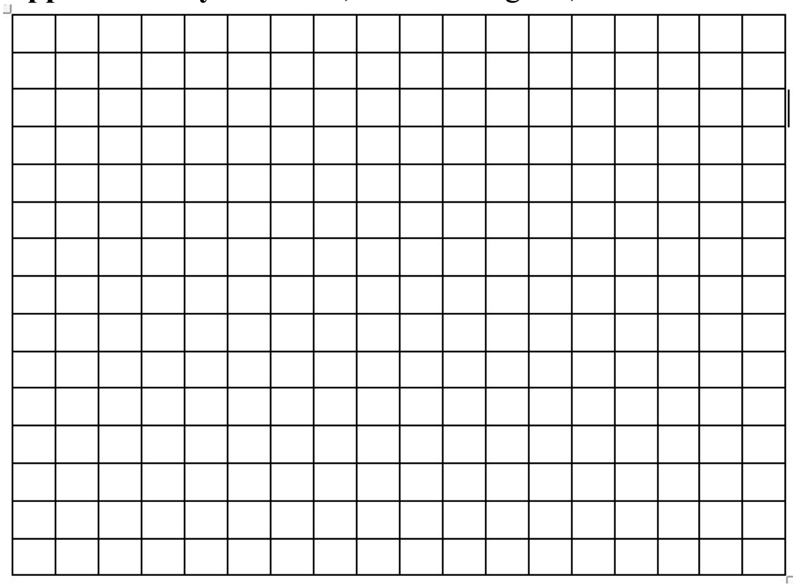
2 rows of 4 squares $2 \times 4 = 8$ $4 \times 2 = 8$

3 rows of 5 squares $3 \times 5 = 15$ $5 \times 3 = 15$

Colour the squares.

Make patterns.

Upper Primary As above, draw more grids, and…



Tell me about (show me) the lattice points?

How many steps are there **up** from one lattice point to the next?

How many steps are there **across** from one lattice point to the next?

Tell me about any number patterns you have noticed?

Is there a connection between the number pattern and the grid size?

Well done, you have found a lattice point on the line! Can you use that information to find another one?

Lower Secondary Some of the above and…

Carefully plot the graph of $3x + 4y = 59$.

One possible method is to find and join the intercepts with the x-axis and y-axis.

Notice that, when $x = 0$, $y = 59/4 = 14\frac{3}{4}$ and when $y = 0$, $x = 19\frac{2}{3}$.

These are not lattice points but they can be plotted and joined to give the straight line graph.

Then the lattice points can be read from the graph and checked numerically.

When you have found the lattice points by this method try to find an alternative method.

Years 9 and 10 Some of the above and…

Find the number of lattice points in the first quadrant (where both x and y coordinates are positive) that lie on the line $3x + 4y = 59$ by at least two methods (there are 4 different methods).

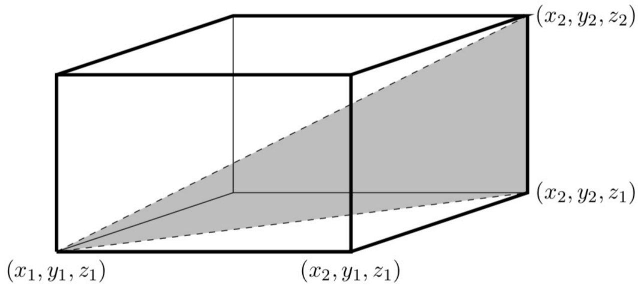
Say which you think is the best method and why.

Use that method to find the lattice points in the first quadrant for $2x + 5y = 52$?

Write a list of instructions for another learner to follow to use that same method for other **examples.**

Years 11, 12 and 13 As for Years 9 and 10 and …

Explain, and write a formal proof, that the formula used in Analytic Geometry for the distance between two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is simply Pythagoras Theorem.



Prove the distance formula in 3 dimensions.

For ages 14 to 16:

You can use this activity to introduce the concept of gradient and the Diagnostic Quiz at the end of your session. Alternatively, you can use this activity to review the concept of gradient and start with the Diagnostic Quiz to find out how much your learners remember and understand about gradients.

Draw a grid and make sure that the learners understand what lattice points are. Then, according to the age group and your learning objectives for the lesson, select the activity you want your learners to start with.

If you have a group you could ask learners to work individually for about 10 minutes and then to work with a partner and share ideas. Have they used the same method or different methods? Can they use both methods to find the points? For bigger groups ask pairs to share ideas with pair.

Then involve the whole group in presenting the different methods that they have used. If time review and compare all four methods.

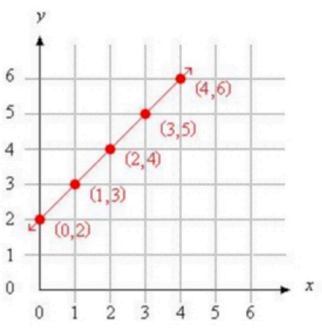
Key questions *These are questions you ask to help learners to* ***think for themselves****.* ***STOP yourself*** *telling children what to do next. Instead ask a* ***KEY QUESTION****.*

- Tell me about (show me) the lattice points?
- Tell me how many steps there are across from one lattice point to the next?
- Tell me how many steps there are up from one lattice point to the next?
- Tell me about any number patterns you have noticed?
- What connection is there between the number pattern and the grid size?
- Well done, you have found a lattice point on the line! Can you use that information to find another one?
- Can you plot a graph of the line?
- Can you find the gradient of the line?
- Does the line go up or down? Is the gradient positive of negative?
- How could you record your working in an organised way?
- Can you think of another method of finding the lattice points?

Diagnostic Assessment This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
"Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".

Which of the following points would continue the pattern of points on the line?



A (5,5)
B (5,6)
C (5,7)
D (5.8)

2.Notice how the learners respond. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. It is important for learners to explain their answers to clarify their thinking by putting it into words, and to develop their communication skills.
4.Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
5.Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.
6.If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

C. is the correct answer.
Common Misconceptions
A. ,**B.** ,**D**. All these answers show a lack of understanding of coordinates and gradients.
https://diagnosticquestions.com

Follow-up ideas

Multiplication tables and gradient: https://aiminghigh.aimssec.ac.za/years-7-9-steps/
Taking readings from a graph: https://aiminghigh.aimssec.ac.za/years-7-10-paper-weight/
Lines and area: https://aiminghigh.aimssec.ac.za/years-8-to-10-graphical-triangle/
Gradients and perpendicular lines: https://aiminghigh.aimssec.ac.za/years-10-to-12-line-match/

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa.
Note: The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is **not** included in the school curriculum for Grade 12 SA.

	Lower Primary or Foundation Phase Age 5 to 9	**Upper Primary Age 9 to 11**	**Lower Secondary Age 11 to 14**	**Upper Secondary Age 15+**
South Africa	**Grades R and 1 to 3**	**Grades 4 to 6**	**Grades 7 to 9**	**Grades 10 to 12**
USA	**Kindergarten and G1 to 3**	**Grades 4 to 6**	**Grades 7 to 9**	**Grades 10 to 12**
UK	**Reception and Years 1 to 3**	**Years 4 to 6**	**Years 7 to 9**	**Years 10 to 13**
East Africa	**Nursery and Primary 1 to 3**	**Primary 4 to 6**	**Secondary 1 to 3**	**Secondary 4 to 6**