TAKE THREE FROM FIVE

Is it always possible, from any set of five positive whole numbers, to choose three numbers that add up to a multiple of 3?
For example, if you start with 4, 7, 11, 12 and 18, you can choose 4+11+12=27 (a multiple of 3).
Try some examples of your own.
If you think it is always possible can you explain why?

Solution:

Count in 3's: 3, 6, 9, 12… Think about multiples of 3.
Every third number is a multiple of 3.
If we add multiples of 3 we always get another multiple of 3.
Every number is a multiple of 3, or one more than a multiple of 3 or two more than a multiple of three.
So there are **3 types of number according to the remainder when we divide by 3**. This is like having only two types (odd and even) for multiples of 2.
The three types of numbers can be written as 3t, 3t+1 or 3t+2 (where t is any integer).

To show that from 5 numbers we can always choose 3 numbers that add up to a multiple of 3 we focus on the remainders when we divide numbers by 3.

We look at all the cases where we choose 3 numbers from 5.

CASE 1: We can choose 3 numbers of the same type
Example: From 5, 6, 9, 11, 15 choose 6, 9 and 15.
If they are **3 multiples of 3, 3a and 3b and 3c,** then the sum is **3(a + b + c**), a multiple of 3.
If they are 3 **multiples of 3 and one more, 3a+1, 3b+1 and 3c+1,** then the sum is a multiple of 3 and 3 more, **3(a + b + c)+3** so it is a multiple of 3.
If they are 3 **multiples of 3 and two more 3a+2, 3b+2 and 3c+2,** then the sum is a multiple of 3 and 6 more, **3(a + b + c)+6** so it is a multiple of 3.

CASE 2: We can chose 3 numbers of different types
Example: From 5, 7, 9, 11, 14 choose 5, 7 and 9.
That is we can choose a multiple of 3, a multiple of 3 and one more, and a multiple of 3 and two more.
So the sum is a multiple of 3 and 3 more, that is another multiple of 3.
Algebraically the numbers are 3a, (3b+1) and (3c+2) and the sum is 3(a + b + c)+3 which is a multiple of 3.

There are no other cases so we can always choose 3 numbers from 5 that add up to a multiple of 3

Diagnostic Assessment This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

If n is a positive integer which is the odd one out of the following 4 numbers?

7	3n+1	3n + 2	16
A	B	C	D

C. is the correct answer. All the other numbers are one more than a multiple of 3 and belong to the set {1, 4, 7, 11, 16, 19 … }. The algebraic expression 3n + 1 defines all the numbers in this set.

Common Misconceptions

Learners who choose A. B. or D. have not recognised the common property shared by the numbers 7, 3n+1 and 16.

https://diagnosticquestions.com

Why do this activity?

This activity looks like a number task, possibly revision about multiples, but it leads to making a conjecture that something always happens and finding reasons and then a proof. The teacher should not do the proof for the class but rather it is better to come back to the activity more than once so that the learners have time to develop their ideas and find the proof for themselves.

Intended Learning Outcomes

1. Learners will gain practice in logical thinking, making conjectures and finding their own explanations and proofs.
2. Learners will meet algebraic expressions that define sets of numbers such as 2n and 2n+1 for even and odd numbers and 3n, 3n+1 and 3n+2 for numbers modulo 3.

Possible approach

Start with the diagnostic question which introduces the idea of an algebraic expression to define a set of numbers. This problem relates to clock arithmetic which everyone is familiar with in relation to time, days of the week and months of the year. It is best not to tell the class that this is the key to the proof as they will gain a lot more from making the connection themselves.

Next introduce the problem by inviting learners to suggest sets of five whole numbers, then you circle three of them that add up to a multiple of three, and write down the total. Repeat this several times. Ask the learners "What am I doing?"

Don't say anything - let learners work out what is special about the sum of the numbers you select. Suggest that if they know what is going on they may like to choose 5 numbers that **stop you** achieving your aim. At some stage check that they all know what is going on.

Challenge them to offer five numbers that don't include three that add up to a multiple of 3. Allow them time to work on the problem in pairs or small groups, and suggest that they write any sets they find up on the board. Learners may enjoy spotting errors among the suggestions on the board. Allow negative numbers, as long as they will allow you negative multiples of 3 (and zero).

At some stage there may be mutterings that it's impossible. You could say: "Well if you think it's impossible, there must be a reason. If you can find a reason then we'll be sure." Or "OK, but can you prove it? Can you convince me that it's impossible?"

If nobody is making any progress ask: “What do we know about multiples of 3?“ “What else?
Make a list of the answers on the board:

- Every third number is a multiple of 3.
- If we add multiples of 3 we always get another multiple of 3.
- There are **3 types of number according to the remainder when we divide by 3**. This is like having only two types (odd and even) when we consider multiples of 2.
- Every number is a multiple of 3 (say 3n), one more than a multiple of 3 (say 3n+1) or two more than a multiple of 3 (say 3n+2).

Then make a table on the board and, ask learners to suggest 5 numbers and say which columns they go in. After filling in a few rows ask learners to do some more and try to prove it is always possible to choose 3 numbers that add up to a multiple of 3.

5 numbers	3n type	(3n+1) type	(3n+2) type	Numbers adding to a multiple of 3
4, 7, 11, 12, 18	12, 18	4, 7	11	12+4+11=27 etc.
5, 6, 9, 11, 15	6, 9, 15		5, 11	6+9+15=30
5, 8, 11, 12, 13	12	13	5, 8, 11	5+8+11=24
6, 7, 8, 10, 13	6	7, 10, 13	8	7+10+13=30

Key questions

What do we know about multiples of 3? What else?

What can we say about whole numbers that are not multiples of 3?
What happens if you add two or more multiples of 3?

What is the remainder when you divide that number by 3?

Possible extension

What Numbers Can We Make Now? (NRICH 8280) is a suitable follow-up task.

Another challenging extension:
You can guarantee being able to get a multiple of 2 when you select 2 from 3.
You can guarantee being able to get a multiple of 3 when you select 3 from 5.
Can you guarantee being able to get a multiple of 4 when you select 4 from 7?

Possible support

Think about and answer this question first. Select sets of 3 numbers. Your sets will always include two numbers that add up to an even number. Why?

See the NRICH POSTER

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa.
Note: The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is **not** included in the school curriculum for Grade 12 SA.

	Lower Primary or Foundation Phase Age 5 to 9	**Upper Primary Age 9 to 11**	**Lower Secondary Age 11 to 14**	**Upper Secondary Age 15+**
South Africa	**Grades R and 1 to 3**	**Grades 4 to 6**	**Grades 7 to 9**	**Grades 10 to 12**
USA	**Kindergarten and G1 to 3**	**Grades 4 to 6**	**Grades 7 to 9**	**Grades 10 to 12**
UK	**Reception and Years 1 to 3**	**Years 4 to 6**	**Years 7 to 9**	**Years 10 to 13**
East Africa	**Nursery and Primary 1 to 3**	**Primary 4 to 6**	**Secondary 1 to 3**	**Secondary 4 to 6**