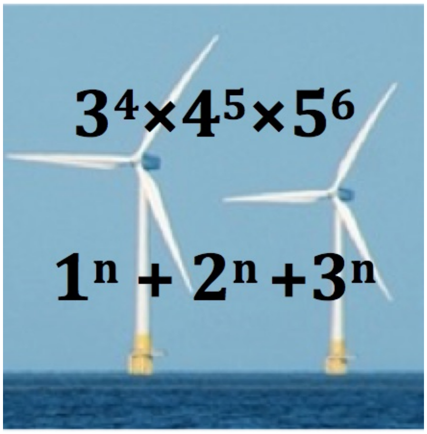
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

POWERFUL THINKING 3



1. Imagine the first number $3^4 \times 4^5 \times 5^6$ is written out in full. How many zeros would there be at the end? Why? Can you find the answer without doing the whole calculation?

2. Work out a few values of the second number $1^n + 2^n + 3^n$. Can you show it will be even for all values of n?

For what values of n will $1^n+2^n+3^n+4^n$ be even?

What about $1^n+2^n+3^n+4^n+5^n$?

What can you say about the sums of powers of the counting numbers 1, 2, 3, 4, 5, 6, 7, …p?

HELP

As with earlier examples the LAST DIGITS of powers repeat in cyclic patterns

$2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$ …. Continue this pattern and the last digits are

2, 4, 8, 6, 2, 4, 8, 6, 2, … and you'll find cyclic patterns like this for other powers.

See Powerful Thinking 1 https://aiminghigh.aimssec.ac.za/grade-7-powerful-thinking-1/
and Powerful Thinking 2 https://aiminghigh.aimssec.ac.za/grades-8-10-powerful-thinking-2/

NEXT

Make up a similar expression involving sums or products of powers of whole numbers and write down a list of the properties. Exchange your work with a partner and check that you agree on the properties and the properties are clearly described.

Notes for teachers

Solution

1. $3^4 \times 4^5 \times 5^6 = 3^4 \text{ x } 2^{10} \text{ x } 5^6$
   $= (3 \text{ x } 2)^4 \text{ x } 2^6 \text{x } 5^6$
   $= (3 \text{ x } 2)^4 \text{ x } 10^6$
   so there will be 6 zeros at the end.

2. $1 + 2 + 3 = 6$, $1^2 + 2^2 + 3^2 = 14$, $1^3 + 2^3 + 3^3 = 36$, $1^4 + 2^4 + 3^4 = 98$ … All these are even.
For all values of n, $1^n = 1$ is odd, 2^n is even and 3^n is always odd. The sum of two odd numbers is always even and the sum of two odds and an even number is even so $1^n + 2^n + 3^n$ is always even for all values of n.

As 4^n is even for all values of n it follows that $1^n+2^n+3^n+4^n$ is even for all values of n.

As 5^n is odd for all values of n it follows that $1^n+2^n+3^n+4^n+5^n$ is odd for all values of n.

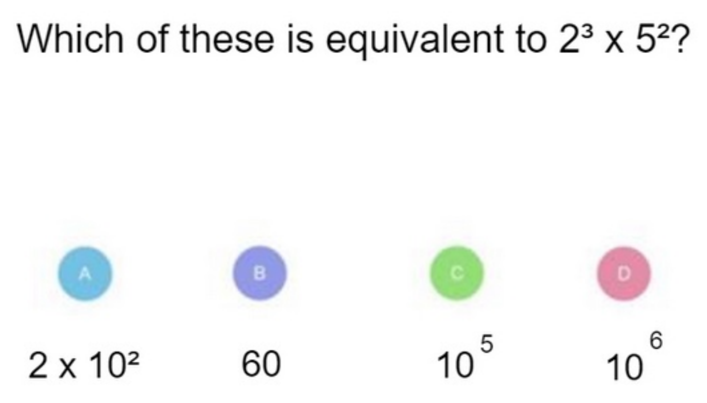
All nth powers of odd numbers are odd and all nth powers of even numbers are even so $1^n + 2^n + 3^n + \ldots + p^n$ is even if p is even and odd if p is odd.

Why do this activity

This activity provides practice for learners in working with exponents using both multiplication and addition, practice in spotting number patterns and in mathematical reasoning and proof. It is the third in a set of similar activities that can be used in the same week or at different times, perhaps one each year, to review and reinforce understanding of the operational rules and concepts relating to exponents.

Diagnostic Assessment This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.



The correct answer is A
$2^3 \times 5^2 = 2 \times 2^2 \times 5^2 = 2 \times (2 \times 5)^2 = 2 \times 10^2$

The other answers show that learners do not understand the notation for exponents.

This diagnostic question leads naturally to part 1 of this activity.

https://diagnosticquestions.com

Learning objectives

In doing this activity students will have an opportunity to:

- develop a better understanding of exponents (otherwise called powers or indices);
- explore cyclic number patterns.

Generic competences (some suggestions, select from list or write your own)

In doing this activity students will have an opportunity to **think flexibly**, be creative and innovative and apply knowledge and skills.

Possible approach

1. Start with the diagnostic quiz which takes a similar form to the first part of this challenge.
2. You might then guide the learners through the first of these activities and then give them the second one to do themselves. Write the number with $3^4 \times 4^5 \times 5^6$ on the board and ask learners to write it in different ways **without multiplying it out and without calculating the product**. They could do this on show boards. Many learners will write out 3 x 3 x… etc. and you will be able to assess their understanding of exponents. If nobody has written 4^5 as 2^{10} you can ask them if they can write 4^5 in a different way without writing it out in full or doing any multiplication. When you think the learners have understood that the given expression is equal to $3^4 \times 2^{10} \times 5^6$ ask them to work out how many zeros would there be at the end without actually doing any multiplication. Give them about 5 or 10 minutes to do this, allowing them to work in pairs and asking for them to write down their reasons. Then ask learners to come to the board to show and explain how they have done it.

3. Write the second question on the board. This question is about sums of powers and odd and even numbers. Suggest the learners start by making lists of the last digit in the powers of 2^n and 3^n for $n = 1$ to 5 and then look for patterns that they can use to answer the question. Emphasise that they do not have to work out the products, simply the last digits. Those who finish can go on to work make up similar patterns involving sums of powers of numbers and noting interesting properties or this could be given for homework. End the lesson with a class discussion where learners explain what the patterns that they have spotted and the mathematical reasoning that they have used to answer the questions.

	1	2	3	4	5	6	7	8	9	10
Discuss the repeating or cyclic patterns in the last digits for n^{th} powers	Last digits for n^{th} powers									
	1	1	1	1	1	1	1	1	1	1
	2	4	8	6	2	4	8	6	2	4
	3	9	7	1	3	9	7	1	3	9
	4	6	4	6	4	6	4	6	4	6
	5	5	5	5	5	5	5	5	5	5

You might ask where the learners have seen repeated cyclic patterns like this before and connect it to division and to recurring decimals, for example 5/7 = 0.7142857… with the digits 714285 repeating over and over again indefinitely.

Key Questions

- Is 4 a prime number? How could you write it as a power of a prime number?
- What do you notice?
- What do you notice about the last digits?
- Can you explain how you know that?
- Can you describe a pattern that you see?
- Is the sum of 2 odd numbers odd or even? How do you know?
- Is the sum of 2 even numbers odd or even? How do you know?
- Is the sum of and odd number and an even numbers odd or even? How do you know?

Follow up

See also Powerful Thinking 4 https://aiminghigh.aimssec.ac.za/grades-10-powerful-thinking-4/

This activity is about prime factorization and writing a number as a product of powers of its factors:
https://aiminghigh.aimssec.ac.za/years-9-11-how_many_factors/

This is a jigsaw puzzle involving powers:
https://aiminghigh.aimssec.ac.za/years-7-10-power-matching/

Finally use this card matching activity for assessment:
https://aiminghigh.aimssec.ac.za/years-9-11-exponents/

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. New material will be added for Secondary 6. The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is beyond the school curriculum for Grade 12 SA.
For resources for teaching A level mathematics see https://nrich.maths.org/12339

	Lower Primary or Foundation Phase Age 5 to 9	Upper Primary Age 9 to 11	Lower Secondary Age 11 to 14	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6