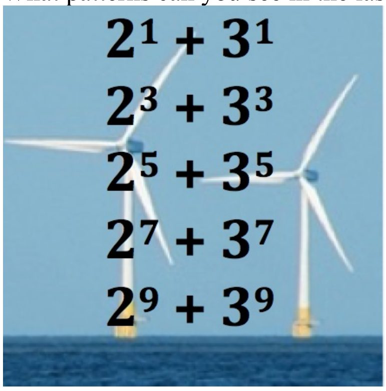
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE
TEACHER NETWORK

POWERFUL THINKING 1

What patterns can you see in the last digits for these numbers?



Can you explain why all these numbers are multiples of 5?

What about the 99th powers $2^{99} + 3^{99}$?

What can you find out about $1^{99} + 2^{99} + 3^{99} + 4^{99}$?

What other patterns with powers can you find that have special properties?

Help

Use the table below. Start filling it in with the $2^3 + 3^3$ box.

$2^3 = 2\times2\times2=8$
$3^3 = 3\times3\times3=27$

Now add $2^3 + 3^3$ and fill the **last digit** in the box.
Work out $2^4 + 3^4$ and write the last digit in the next box.
Carry on in this way.
Look at the last digits for the odd powers in the yellow highlighted columns. What do you notice.

	1	2	3	4	5	6	7	8	9	10
Last digits for the powers given above										
$2^{\text{power above}}$	2	4	8	6	2					
$3^{\text{power above}}$	3	9	7	1	3					
$2^{\text{power above}} + 3^{\text{power above}}$	2^1+3^1 5	2^2+3^2 3	2^3+3^3							

Next

Make up your own similar expression involving sums or products of powers of whole numbers.
Write down a list of the properties that you see.
Perhaps exchange you work with another learner and each of you can check the expressions invented by the other and check that the properties are correctly described.

Notes for teachers

Solution

The table shows the last digits in the powers of 1, 2, 3, …9. The last digits repeat in cycles.

For example the last digits in the powers of 2 are 2,4,8,6,2,4… and the last digits in the powers of 3 are 3,9,7,1,3,9,7… both cycles of length 4.

Adding the sums of the last digits of the numbers with odd powers: $2^1 + 3^1$, $2^3 + 3^3$, … are always 5 **for all odd powers so these numbers are divisible by 5**.

Base number	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10
Last digits for n^{th} powers									
1	1	1	1	1	1	1	1	1	1
2	4	8	6	2	4	8	6	2	4
3	9	7	1	3	9	7	1	3	9
4	6	4	6	4	6	4	6	4	6
5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6
7	9	3	1	7	9	3	1	7	9
8	4	2	6	8	4	2	6	8	4
9	1	9	1	9	1	9	1	9	1

As 99 is an odd number we know that $2^{99} + 3^{99}$ is divisible by 5 (is a multiple of 5) because it has 5 as the last digit.

The last digits in the powers of 4 repeat in a 2-cycle 4, 6, 4, 6,… with the last digit for odd powers always equal to 4.

Also 1 raised to any power equals 1 so the last digit of $1^{99} + 2^{99} + 3^{99} + 4^{99}$ ends in 0 and this number is divisible by 10 (also by 5).

Diagnostic Assessment

This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

Calculate

$4^2 + 2^3$

A 20
B 24
C 22
D Do not know

The correct answer is B $4^2 + 2^3 = 16 + 8 = 24$

A. The learner may have worked out $4^2 + 2^2$

C. Might be a guess

https://diagnosticquestions.com

Why do this activity?

This activity provides practice for learners in working with exponents using both multiplication and addition, also practice in spotting number patterns and in mathematical reasoning and proof. It provides several related activities that cater well for all abilities. There are extension questions for gifted learners and the open ended nature of the questions allow learners to be creative in their thinking. Younger or slower learners can work entirely with the numbers. Older learners can be asked to make general algebraic statements to describe the patterns like $2^n + 3^n$ is divisible by 5 for all even values of n and $a^n + b^n$ is divisible by 5 for all odd values of n for all integers a and b such that (a + b) is divisible by 5.

It is the first in a set of similar activities that can be used in the same week or at different times, perhaps one each year, to review and reinforce understanding of the operational rules and concepts relating to exponents. You could use it to assess prior knowledge of exponents before moving on to further work. It provides an opportunity to use the vocabulary of divisibility and multiples and factors as well as the different words for exponents (powers and indices). Learners may enjoy the beauty of these numerical patterns and want to find some more patterns for themselves.

Learning objectives

In doing this activity students will have an opportunity to:

- practise calculating powers of 2 and 3
- practise looking for number patterns and trying to explain them.

Generic competences

In doing this activity students will have an opportunity to:

- **think flexibly**, be creative and innovative and apply knowledge and skills;
- **persevere and work systematically** to investigate and explain patterns.

Suggestions for teaching

This question asks about patterns in the last digits of numbers and you might give it to your class to work on individually or in pairs, and then perhaps in groups of 4. Quicker workers who finish the first part can go on to 99th powers but when most of the class has answered the first part of the question you could have a class discussion of what the learners have discovered and about the repeating cyclic patterns in the last digits, before the learners all proceed with the 99th powers and the rest of the question.

If your class needs a lot more support then you might like to ask the learners to write down the last digits in the powers of 2 for 2, 2^2, 2^3, 2^4 and 2^5 without working out the whole product to be sure that they understand that they are focusing on the last digits. Then ask them to write down the last digits in the remaining powers of 2 up to 2^9 and also in the powers of 3 up to 3^9 and to answer the question about $2^1 + 3^1$, $2^3 + 3^3$ … etc .

You might help the learners to record the last digits in a table similar to the table given above and then discuss the repeating cyclic sequences.

You might ask where the learners have seen repeated cyclic patterns like this before and connect it to division and to recurring decimals, for example 4/11 = 0.36363… with the digits 36 repeating over and over again indefinitely and 2/7 = 0.285714285… with the digits 285714 repeating over and over again indefinitely.

Key Questions

What do you notice?
Can you describe a pattern that you see?
What is special about the powers 1, 3, 5, 7 and 9 in this question?
What do you notice about the last digits?
Can you explain how you know that?
How do you know when a number is divisible by 5?
How do you know when a number is a multiple of 5?
Can make up your own pattern of numbers involving sums of powers that share the same property?

Follow up

See also Powerful Thinking 2 https://aiminghigh.aimssec.ac.za/years-8-10-powerful-thinking-2/
Powerful Thinking 3 https://aiminghigh.aimssec.ac.za/years-9-10-powerful-thinking-3/
and Powerful Thinking 4 https://aiminghigh.aimssec.ac.za/years-10-12-powerful-thinking-4/

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA and to Years 4 to 12 in the UK.

	Lower Primary or Foundation Phase	Upper Primary	Lower Secondary	Upper Secondary
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6