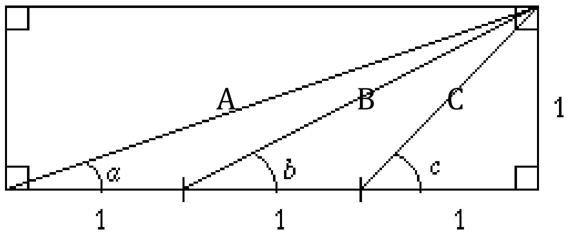
Title: THREE BY ONE



This rectangle measures three units by one unit.
The diagram shows three angles a, b and c.
How many ways can you prove that a + b = c?
What other interesting properties can you find?

Solution

Here are a few of the many different methods of solution.

Method 1 Tan Angle Sum Formula

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a . \tan b} = \frac{1/3 + 1/2}{1 - 1/6} = \frac{5/6}{5/6} = 1 = \tan c \qquad \text{so } \underline{a + b = c}$$

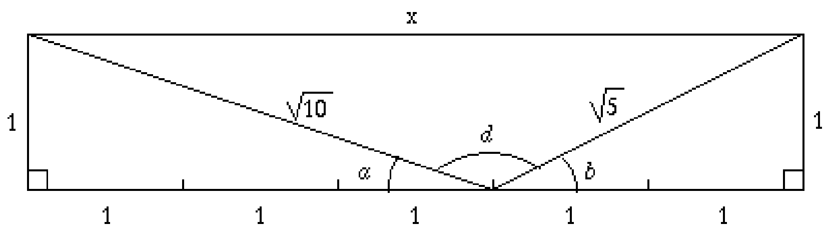
Method 2 Sin Angle Sum Formula

$A = \sqrt{10}$, $B = \sqrt{5}$ and $C = \sqrt{2}$

$$\begin{aligned}\sin(a+b) &= \sin a \cos b + \cos a \sin b \\ &= 1/\sqrt{10} . 2/\sqrt{5} + 3/\sqrt{10} . 1/\sqrt{5} \\ &= 2/5\sqrt{2} + 3/5\sqrt{2} \\ &= 5/5\sqrt{2} = 1/\sqrt{2} \\ &= \sin c \text{ so}\end{aligned}$$

$\underline{a + b = c}$

Method 3 Cosine Rule



This diagram is an extension of the original diagram and shows that $x = 3 + 2 = 5$ and $a + b = 180 - d$

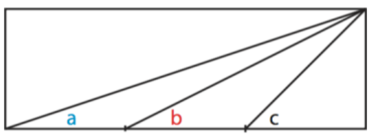
By the cosine rule

$$\cos d = \frac{10 + 5 - 25}{2\sqrt{10}.\sqrt{5}} = \frac{-10}{10\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

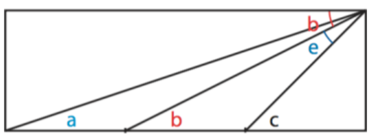
So $d = 135^o$

It follows that $a + b = 180 - 135 = 45^o$ and so $\underline{a + b = c}$

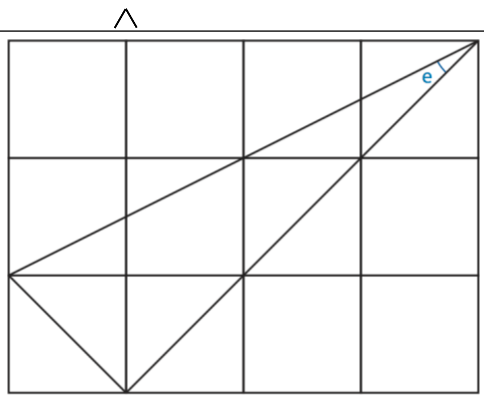
Method 4 Geometry



The two angles marked b are equal by alternate angles.



c = e + b as c is the exterior angle to the triangle in which a and e are interior angles.



By extending the grid we see that this triangle is right angled and $e = \tan^{-1}1/3 = a$

We have proved
c = e + b so
$\underline{c = a + b}$

Note $\tan^{-1}$ is also written arctan.

Another interesting property concerns the lengths of the hypotenuses of the triangles. We prove A = BC

$\underline{\mathbf{A = \sqrt{10} = \sqrt{5}.\sqrt{2} = BC}}$

Notes for teachers

Why do this activity?

This activity exemplifies the unity of mathematics and that almost all the mathematics we learn is connected to the rest of what we learn. It is a good none-routine exercise that can engage learners in thinking for themselves about trigonometry and geometry and giving them an opportunity to discover and compare different methods of solving the same problem.

You might use it as a challenging activity for high flyers who are doing well in trigonometry or you might use it for a whole class in a revision session.

Possible approach

You could use this problem for revision. Divide the class into four groups and start them working, individually at first, telling one group to use the Tan Angle Sum Formula to prove a = b + c, another group to use the Sine Angle Sum Formula, another group to use the Cosine Rule and another group to use Geometry. If you don't want this work to take much time you could give the Cosine Rule group and the Geometry group the diagrams from the solution given here.

You could use the 'One, Two, Four, More' strategy, suggesting the learners work individually, then at a signal from you, in pairs and later in fours.

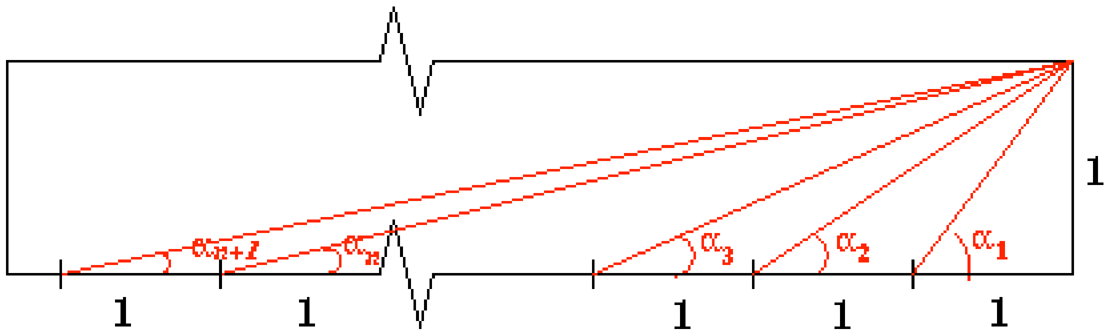
Tell them that you are going to ask one pair from each group to explain their method to the whole class writing their solution on the chalkboard.

Key Questions

- Do you see anything interesting in the diagram?
- What are the tangents of the angles?
- Can you find the lengths of the hypotenuses?
- What are the sines of the angles?
- Can you see the connection between that diagram and the original diagram.

Possible Extension

Given a rectangle of dimensions 1 by n, for each angle, αm where m is a positive integer, are there two other angles αp and αq whose sum is equal to αm?



Toni first published the problem 'Three by One' on the NRICH website in June 1998 and two school students from Scotland, Alex Goodwin and Neil Donaldson not only submitted 8 different methods for proving a + b = c but also suggested this extension and sent in a very nice proof. See http://nrich.maths.org/1334

Possible Support

If a student is really struggling, and you want him to succeed, you could suggest that all he has to do is to find tan(a + b) using the formula, and to find tan c, and to show that they are equal.