AIMSSEC

AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

WHY 24?

1. Write down three consecutive whole numbers none of which is a multiple of three. If you can't do it, explain why.

2. Multiply any two consecutive even numbers together. Why is the product always a multiple of 8?

3. Take any prime number greater than 3, square it, subtract one and divide by 24.

Make a statement about what you notice about squaring prime numbers, subtracting one and dividing by 24 (a conjecture) and prove that what you say is always true.

Help

You have to make a statement about what you notice about squaring prime numbers, subtracting one and dividing by 24 (a conjecture) and prove that **what you say is true for all prime numbers greater than 3**.

FIRST CARD IN PROOF Let p be a prime number greater than 3.	*The first and last cards are in the right places. Rearrange the other cards so that each statement follows from the one before it.*
p is an odd number, so p - 1 and p + 1 must both be multiples of 2.	(p - 1)(p + 1) is the product of a multiple of 2 and a multiple of 4, so must be a multiple of 8.
(p - 1) and (p + 1) are consecutive even numbers so either (p - 1) or (p + 1) must be a multiple of 4.	(p - 1), p, and (p + 1) are consecutive numbers.
p is prime and greater than 3 so cannot be a multiple of 3.	(p - 1)(p + 1) is a multiple of both 8 and 3, so (p - 1)(p + 1) is a multiple of 24.
The expression $p^2 - 1$ can be factorised as (p - 1)(p + 1)	Either (p - 1) or (p + 1) must be a multiple of 3, so the product (p - 1)(p + 1) must be a multiple of 3.
If I have three consecutive numbers, one of them must be a multiple of 3.	**LAST CARD IN PROOF** Therefore for any prime number p greater than 3, $p^2 - 1$ is a multiple of 24.

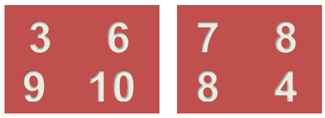
Try it for the prime number 7 and you will find $7^2 - 1 = 49 - 1 = 48$ which is a multiple of 24. What about $11^2 - 1$? What about $13^2 - 1$?

This is a Proof Sorter Activity. If possible work as a group and do not move to the next step in the proof until everyone understands everything up to that step.

Cut out the larger cards on the next page and share them between the members of the group. Each person should study their card(s) and then explain to everyone else what it means. Use different values of p like p = 7 or 11 or 13 or 17 to make sense of the statements and convince everyone in the group that the statement on the card is true for all prime numbers p greater than 3.

Arrange the cards in order to give a proof of the conjecture.

Next



Write down your own explanation of the proof of the conjecture.

How many ways can you combine 4 whole numbers from 1, 2, 3 … 10 by addition, subtraction, multiplication and division to get the answer 24? Numbers can be repeated. For the numbers 4, 7, 8, 8, a solution is $(7 - (8 \div 8)) \times 4 = 24$ and for 3, 6, 9 and 10 here are 2 solutions: $6 \times (3+10-9) = 24$; $(6\times9) - (3\times10) = 24$.

Make your own 24 Game. Make lots of cards, each with 4 numbers from 1 to 10 that can be combined by +, -, × and ÷ to get the answer 24. To play the game use a timer, give one card to the first player. S/he keeps the card if she can make 24, and if not the first of the other players to make 24 gets the card, or it is shuffled back into the pack if nobody can make 24. When all the cards have been used the player with most cards wins the game. Alternatively, use an ordinary pack of playing cards, make aces take the value 1 and honour cards the value 10. Deal 4 cards to the first player and proceed as before.

FIRST CARD IN PROOF Let p be a prime number greater than 3.	*The first and last cards are in the right places. Rearrange the other cards so that each statement follows from the one before it.*
p is an odd number, so p - 1 and p + 1 must both be multiples of 2.	(p - 1)(p + 1) is the product of a multiple of 2 and a multiple of 4, so must be a multiple of 8.
(p - 1) and (p + 1) are consecutive even numbers so either (p - 1) or (p + 1) must be a multiple of 4.	(p - 1), p, and (p + 1) are consecutive numbers.
p is prime and greater than 3 so cannot be a multiple of 3.	(p - 1)(p + 1) is a multiple of both 8 and 3, so (p - 1)(p + 1) is a multiple of 24.
The expression $p^2 - 1$ can be factorised as (p - 1)(p + 1)	Either (p - 1) or (p + 1) must be a multiple of 3, so the product (p - 1)(p + 1) must be a multiple of 3.
If I have three consecutive numbers, one of them must be a multiple of 3.	**LAST CARD IN PROOF** Therefore for any prime number p greater than 3, $p^2 - 1$ is a multiple of 24.

NOTES FOR TEACHERS

SOLUTION

Every third whole number must be a multiple of 3 so it is impossible to write down three consecutive whole numbers none of which is a multiple of three. One of the 3 consecutive whole numbers will always be a multiple of 3.

Alternate even numbers are multiples of 2 and 4 so the product of 2 consecutive even numbers is a multiple of 8.

If n is even then 2n is a multiple of 4 and 2n+2 is a multiple of 2.

If n is odd then 2n is a multiple of 2 and 2n+2 is a multiple of 4.

If p is a prime number then $p^2 - 1 = (p - 1)(p + 1)$ and both $(p - 1)$ and $(p + 1)$ are even. Also, because p is prime it is not a multiple of 3, so either $p - 1$ or $p + 1$ must be a multiple of 3.

So for all prime numbers greater than 3 the number $p^2 - 1$ is a multiple of $3 \times 8 = 24$.

Diagnostic Assessment

This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

In the following list of numbers, how many are prime?

1, 2, 4, 5, 9, 31, 33

A. 2
B. 3
C. 4
D. 5

The correct answer is B as 2, 5 and 31 are prime numbers.

A. and D. May be guesses

C. Learners giving this answer make have the misconception that 1 is a prime number.

https://diagnosticquestions.com

Why do this activity?

This activity is well suited for learners who are working on the difference of two squares. It involves a significant 'final challenge' which can either be tackled on its own or after working on a set of related 'building blocks' designed to lead students to helpful insights.

The problem is structured in a way that makes it ideal for learners to work on in small groups.

Learning Objectives

- To review multiples and factors.
- To experience making a general conjecture based on the results of numerical examples.
- To experience using knowledge of the difference of two squares, and of factors and multiples, to prove a general result.

Generic competences

In doing this activity students will have an opportunity to:

- **think mathematically,** reason logically and give explanations and proofs;
- **work in a team**:
  - collaborate and work with a partner or group
  - have empathy with others, listen to different points of view
  - develop leadership qualities;
- **communicate** in writing, speaking and listening:
  - exchange ideas, criticise, and present information and ideas to others
  - analyze, reason and record ideas effectively.

Possible approach

This task might ideally be done in groups of four. Each learner could be given either 1 or 2 below as a building block to work on.

1. Write down three consecutive whole numbers none of which is a multiple of three.
If you can't do it, explain why.

2. Multiply any two consecutive even numbers together. Why is the product always a multiple of 8?

After they have had an opportunity to make progress individually on their question, encourage them to share their findings with each other and work together on each other's tasks.

When everyone is satisfied that they have explored in detail challenges 1 and 2, hand out the final challenge, building block number 3.

3. Take any prime number greater than 3, square it, subtract one and divide by 24.

Make a statement about what you notice about squaring prime numbers, subtracting one and dividing by 24 (a conjecture) and prove that what you say is always true.

The teacher's role is to challenge groups to explain and justify their mathematical thinking, so that all members of the group are in a position to contribute to the solution of the challenge.

Alternatively, for some or all of the class, the teacher might give out the cards on page 2 as a different set of building blocks. This is a Proof Sorter activity. The challenge is to arrange the cards in order to give the proof of the conjecture. See the **Help** box on page 1.

It is important to set aside some time at the end for learners to share and compare their findings and explanations, whether through discussion or by providing a written record of what they did.

Key questions

- What important mathematical insights does my building block give me?
- How can these insights help the group tackle the final challenge?
- Think of any prime greater than 3. Work out your prime squared minus 1. Can you factorise the answer?
- Work out your prime squared minus 1. What happens when you divide the answer by 24?
- Can you explain why the same result happens for all primes?

Follow up

Proofs of the Difference of Two Squares formula:
https://aiminghigh.aimssec.ac.za/years-8-10-difference-of-squares-and-area/
https://aiminghigh.aimssec.ac.za/years-9-12-differences-of-squares-investigation/

Also play **the 24 Game**. See the **Next** box on page 1.

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is beyond the school curriculum for Grade 12 SA. New material will be added for Y13 and Secondary 6.
For resources for teaching A level mathematics see https://nrich.maths.org/12339

	Lower Primary or Foundation Phase Age 5 to 9	Upper Primary Age 9 to 11	Lower Secondary Age 11 to 14	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6