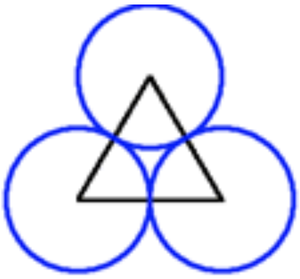
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE
TEACHER NETWORK

POLYCIRCLES



A triangle has sides 12, 13 and 15 units. Circles are drawn with centres at the vertices of the triangle so that each circle touches the other two.

Find the radii of the circles.

Can you always find touching circles for any triangle?

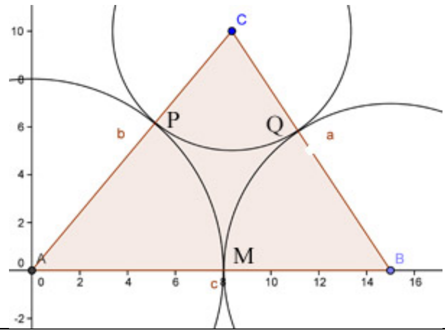
What about other polygons?

SOLUTION

If the 3 circles have radii a, b and c then

$a + b = 12$ (1)
$b + c = 13$ (2)
$c + a = 15$ (3)
From (1) and (2) $c – a = 1$ (4)
From (3) and (4) $c = 8$ and $a = 7$
So $b = 5$.
The radii are 5, 7 and 8 units.

Using the method above if
$a + b = p$
$b + c = q$
$c + a = r$
then the solutions are: $\frac{1}{2}(p - q + r)$, $\frac{1}{2}(p + q – r)$ and $\frac{1}{2}(- p + q + r)$.
Equations like this will always have a solution so you can find touching circles for any triangle.



If you have access to Geogebra then learners can experiment to find solutions. They can use Geogebra to demonstrate that once they have found one solution then the shape and size of the triangle can be changed and the circles change correspondingly.

NOTES FOR TEACHERS

Why do this activity?

This is a rich activity that is accessible and provides suitable challenges for different age groups. Younger learners can find solutions for three circles, given the sides of a triangle, by trial and improvement.

Once the learners can solve two simultaneous linear equations in two unknowns then they can be challenged to solve this problem by solving three simultaneous equations. Because the coefficients are all unity the equations are easy to solve.

The activity demonstrates a powerful inter-relationship between geometry and algebra.

Intended Learning Outcomes

To develop confidence in problem solving through learners moving from trial and improvement methods to algebra and to develop confidence in solving simultaneous linear equations**.**

Possible approaches

Younger and less confident learners may start with trial and improvement methods.

For learners who know how to solve a pair of simultaneous linear equations this problem provides a good series of challenges. Taking a numerical example where the lengths of the sides of the triangle are known, and the radii of the polycircles has to be found, then three simultaneous linear equations can easily be found and solved. Even though learners may only have been taught to solve two simultaneous equations in two unknowns many will be able to solve these three equations for themselves and get satisfaction from being able to do so independently. The results can be checked by drawing.

The next step is to generalise from a particular numerical example and to use exactly the same steps in the algebra to derive formulae for the radii in terms of the lengths of the sides of the triangle. This is a good exercise in a concrete setting for working with algebra.

Learners who have succeeded so far can explore the existence of polycircles for quadrilaterals and pentagons and how the algebra explains the different geometrical phenomena that arise.

For older learners who have been introduced to linear algebra, the generalisation of the problem to polygons with n sides provides a challenge to explain why there are unique solutions for certain values of n and not for others.

Possible support

The problem is written to start with a numerical special case of a triangle in order to support younger and less confident learners. They may first try trial and improvement where the problem is like arithmagons. From that they may be able to develop a method for finding solutions in the general case and even for getting the formula without the use of simultaneous equations.

Possible extension

You could pose the problem for general polygons of 4, 5, 6, …n sides and leave the learners to decide for themselves whether or not to start with special cases.

See the next page for a general solution for an n sided polygon.

Now consider a convex polygon with n sides: a_1 , a_2 , a_3 ,...a_n .
Suppose circles can be drawn with centres at the vertices of the polygon such that the circles just touch each other. Let r_1, r_2, r_3,... be the radii. Then:

$r_1 + r_2 = a_1$

$r_2 + r_3 = a_2$

...

$r_n + r_1 = a_n$

Let's try to figure out what $r1$ equals, and all the other solutions will of course be symmetrical.

Solving these equations:

$$r_1 = a_n - r_n = a_n - a_{n-1} + r_{n-1}$$
$$= a_n - a_{n-1} + a_{n-2} - r_{n-2}$$
$$= a_n - a_{n-1} + a_{n-2} - \ldots +(-1)^{n-1}a_1 + (-1)^n r_1$$

If n is even then $r_1 = a_n - a_{n-1} + a_{n-2} - \ldots - a_1 + r_1$ which means that the condition for the existence of solutions is: $a_n - a_{n-1} + a_{n-2} - \ldots - a_1 = 0.$

It also means that if this condition holds there are an infinite number of solutions and if it does not hold there are no solutions.

If n is odd then what we get is $r_1 = a_n - a_{n-1} + a_{n-2} - \ldots + a_1 - r_1$

so $r_1 = ½(a_n - a_{n-1} + a_{n-2} - \ldots + a_1)$

and r_1 is positive when $a_n - a_{n-1} + a_{n-2} - \ldots + a_1 > 0.$

For polygons with an odd number of sides solutions always exist but 'negative' values of r_i occur when, instead of touching externally, one circle surrounds its 'neighbour' which touches it internally so the length of the edge is given by the difference of the radii and not the sum of the radii.